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Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
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Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 7

(To be handed in on 30. November and discussed on 3. and 4. December)

7.1 Alternative representations of the Dirac matrices (5 + 2 + 2 + 4 Points)

Consider the Dirac equation

$$(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar})\psi_D = 0,$$

where γ^μ are the Dirac matrices in the standard representation.

a) Construct a $U \in SU(4)$ such that the two-component *Weyl spinors* ψ, $\widehat{\psi}$ solve the equation

$$\begin{pmatrix} 0 & W \\ \widehat{W} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} + \frac{mc}{\hbar} \begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} = 0, \tag{1}$$

if $\psi_D = U\binom{\psi}{\widehat{\psi}}$ solves the Dirac equation. W, $\widehat{W}$ are the *Weyl operators*:

$$W = \mathbb{1}_2 \partial_0 - \vec{\sigma} \cdot \vec{\nabla}, \qquad \widehat{W} = -\mathbb{1}_2 \partial_0 - \vec{\sigma} \cdot \vec{\nabla}$$

A real representation of the Dirac equation was found by E. Majorana in 1937. We now consider the unitary transformation U that generates other representations of the γ-matrices from the standard representation.

b) Show that the matrix $U = \frac{1}{\sqrt{2}}\gamma^0(\gamma^2 + \mathbb{1}_4)$ is unitary and that $U^2 = \mathbb{1}_4$.

c) Let $D = i\gamma^\mu \partial_\mu$ be the Dirac operator: . Show that $D' = UDU$ is real and that $\psi' = U\psi$ solves the equation $D'\psi' = \frac{mc}{\hbar}\psi'$, if ψ is a solution of the Dirac equation $D\psi = \frac{mc}{\hbar}\psi$.

d) *Particular representations of the Majorana matrices*
You have seen a particular representation of the Majorana γ matrices in the lecture; here we show that we can generate another representation of the Majorana γ-matrices by the transformation

$$\gamma_M^\mu = U\gamma^\mu U^\dagger. \tag{2}$$

Hence show that

$$\gamma_M^0 = \gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma_M^1 = \gamma^2\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

$$\gamma_M^2 = -\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma_M^3 = \gamma^2\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

and compare these with those you have seen from the lecture.

7.2 The Weyl Equation $(3+3+3+3$ Points$)$

We now consider fermionic particles with rest mass $m = 0$.

a) For the Weyl operators W, $\widehat{W}$ defined in the previous exercise show $W\widehat{W} = \widehat{W}W = -g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\Box$ und deduce that each spinor component of the solutions of the Weyl equations satisfy the Klein-Gordon equation with mass $m = 0$.

b) Let P be the parity operator. Show that $P^{-1}WP = -\widehat{W}$, i.e. the Weyl operators are not invariant under parity.

c) Use the ansatz $\widehat{\psi} = \widehat{\mu}_0 e^{-\frac{i}{\hbar}p_\mu x^\mu}$, where $\widehat{\mu}_0$ is a simple scalar constant to prove that $p_\mu p^\mu = 0$. In addition show that for positive energy spin and momentum are antiparallel (negative helicity), but parallel for negative energy.

d) Perform a similar analysis as in 7.2c) for the ansatz $\psi = \mu_0 e^{-\frac{i}{\hbar}p_\mu x^\mu}$. What about the directions of spin and momentum? What is the connection between the solution of the two Weyl equations? Which elementary particle do they describe in good approximation?

If one wants to introduce a mass m in the Weyl representation, one will have to consider the coupled system of equation (1), which was shown to be equivalent to the standard Dirac equation in exercise 7.1.

7.3 Properties of the Majorana Representation $(3+3$ Points$)$

The Majorana representation has been defined in the lecture. It has the property that $\psi_M^C = \psi_M^*$, where here ψ_M^C is the solution related to ψ_M by charge conjugation (the subscript M stands for Majorana). This implies, physically, that a Majorana particle is its own antiparticle. We note that a *Majorana spinor* is a spinor that is real, i.e.,

$$\psi_M^* = \psi_M \tag{3}$$

a) *Connection between the chiral and the Majorana representations*
We have seen in the lecture and in exercise 7.2 that in the chiral representation and in the case of particles with rest mass $m = 0$ the Weyl spinors ψ, $\widehat{\psi}$ obey a set of 2 *independent* equations called the Weyl equations. Let us now impose the property of the Majorana spinors $\psi_M^C = \psi_M^*$ and the realness condition, Eq. (3), on the Weyl spinors $\Psi \equiv \binom{\psi}{\widehat{\psi}}$, i.e., set $\Psi^C = \Psi^*$ and $\Psi^* = \Psi$, and show that this recouples the Weyl spinors ψ and $\widehat{\psi}$.
Hint: Use the definition of the charge conjugation operator $\mathcal{C} = i\gamma_W^2 K_0$, where K_0 is the complex conjugation operator and γ_W^2 is in the chiral representation, as seen in the lecture.

b) *Lorentz invariance of the Majorana condition*
In order for the Majorana condition Eq. (3) to be physically meaningful it needs to hold irrespective of the reference frame, that is, is Lorentz invariant. To do this, we know from the lecture that an arbitrary spinor $\Psi(x)$ transforms as

$$\Psi'(x') = \exp(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu})\Psi(x) \tag{4}$$

where the same notation from the lecture has been used. Using the particular properties of the Majorana spinor $\Psi_M(x)$, show that it transforms in the same way as in Eq. (4)