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Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
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Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 6

(To be handed in on 23. November and discussed on 26. and 27. November)

6.1 Spin Expectation Value for a Free Dirac Particle (1 + 3 Points)

We consider a free Dirac particle with energy E and momentum $\vec{p} = (p, 0, 0)$ which moves in the x-direction. The wavefunction of the particle in the rest and laboratory frame reads

$$\Psi_{\vec{p}=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \text{and} \qquad \Psi(\underline{x}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p}{E+m} \end{pmatrix} e^{-ip_\mu x^\mu} \tag{1}$$

respectively.

a) Compute the normalization factor $N = \frac{1}{V} \int d^3x \; \Psi^\dagger \Psi$.

b) Calculate the *spin expectation value* in the rest frame as well as in the laboratory frame,

$$\langle \sigma_z \rangle = \frac{1}{N} \frac{1}{V} \int d^3x \; \Psi^\dagger \Sigma_z \Psi. \tag{2}$$

What happens to this quantity in the relativistic limit? Explain this result physically.

6.2 Dirac Particle in a Potential Well (5 + 2 Points)

We consider the Dirac equation in one dimension in the presence of the following symmetric electro-

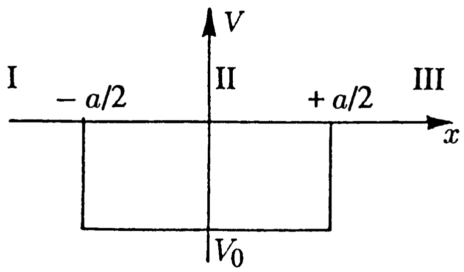

Figure 1: A square well potential with the respective regions I, II and III

static square well potential, as depicted in Fig. 1.

$$V(x) = \begin{cases} 0 & |x| > a/2, \\ -V_0 < 0 & |x| \leq a/2. \end{cases} \tag{3}$$

a) Calculate the solution of the Dirac equation for this potential. To do this, assume outgoing (relative to the potential well) solutions with real or imaginary momentum outside the well and a superposition of left- and right propagating solutions inside it. In addition, use the condition that the wavefunctions must be continuous at the boundaries of the potential well, and hence determine the coefficients of the Dirac spinors in- and outside the well.

b) Discuss the behavior of the solutions for different well-depths and widths. For this question you should differentiate between the case when the deepest bound state (with energy E_0) stays above the Dirac sea ($E_0 > -mc^2$) and where it "dives" below it ($E_0 < -mc^2$).
Remark: The problem described in this exercise can be used as a simplified model of a Dirac electron bound to a heavy atomic nucleus. When the potential well "dives" into the Dirac sea, electron-positron pairs could be spontaneously produced, with the electron remaining bound inside the well, while the positron propagates away. An experiment of this form has been performed at the GSI Darmstadt, in which a heavy nucleus was produced by the collision of two lighter nuclei.

6.3 Zitterbewegung $(2+5$ Points)

We have seen in the lecture that the presence of negative energy solutions of the Dirac equation yields interesting physical phenomena, an example of which is the Klein paradox. In this exercise we will investigate another such phenomenon which again has no counterpart in nonrelativistic quantum mechanics. This phenomenon manifests itself as an oscillatory motion of the Dirac wavepacket which is superimposed on the usual motion with constant velocity. We will see in the following that this stems from the interference between the positive and negative energy solutions.
We consider a general spinor consisting of a superposition of positive and negative energy solutions, as introduced in the lecture:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{E}\sum_{r=1,2}\left(b(p,r)u_r(p)e^{-ipx} + d^*(p,r)v_r(p)e^{ipx}\right) \tag{4}$$

Here, $u_r(p)$ and $v_r(p)$ are spinors for free particles moving in x-direction with positive and negative energies, respectively, while $b(p,r)$ and $d(p,r)$ are complex amplitudes. We note that x and p are 4-component vectors, and the product in the exponents should be understood as an invariant scalar product.

a) We define the expectation value of the position operator as $\langle\vec{x}\rangle \equiv \int d^3x\, \psi^\dagger(x)\vec{x}\psi(x)$. This quantity represents the *amplitude* of the *Zitterbewegung* motion that we are considering. Calculate first the temporal variation $\frac{d}{dt}\langle\vec{x}\rangle$.
Hint: $\psi(t,\vec{x}) = e^{-iHt}\psi(0,\vec{x})$, and use the canonical form of the Dirac Hamiltonian $H = \vec{\alpha}\cdot\frac{1}{i}\vec{\nabla} + \beta m$

b) Integrate the expression you obtained in (a) to to get the following expression for the expectation value of the position $\langle x^i\rangle$ (x^i means the ith component of the position vector $\vec{x}$):

$$\begin{aligned}\langle x^i\rangle = \langle x^i\rangle_{t=0} &+ \int \frac{d^3p}{(2\pi)^3}\frac{mp^i}{E^2}\sum_r\left[|b(p,r)|^2 + |d(p,r)|^2\right]t \\ &+ \sum_{r,r'}\int\frac{d^3p}{(2\pi)^3}\frac{m}{2E^2}\left[b^*(\bar{p},r)d^*(p,r')e^{2iEt}\bar{u}_r(\bar{p})\sigma^{i0}v_{r'}(p) + b(\tilde{p},r)d(p,r')e^{-2iEt}\bar{v}_{r'}(p)\sigma^{i0}u_r(\tilde{p})\right]\end{aligned}$$

Hint: Make use of the identities

$$\begin{aligned}\bar{u}_r(p)\gamma^\mu u_{r'}(q) &= \frac{1}{2m}\bar{u}_r(p)\left[(p+q)^\mu + i\sigma^{\mu\nu}(p-q)_\nu\right]u_{r'}(q)\\ \bar{v}_r(p)\gamma^\mu v_{r'}(q) &= -\frac{1}{2m}\bar{v}_r(p)\left[(p+q)^\mu + i\sigma^{\mu\nu}(p-q)_\nu\right]v_{r'}(q)\\ \bar{u}_r(p)\gamma^\mu v_{r'}(q) &= \frac{1}{2m}\bar{u}_r(p)\left[(p-q)^\mu + i\sigma^{\mu\nu}(p+q)_\nu\right]v_{r'}(q)\\ \bar{v}_r(p)\gamma^\mu u_{r'}(q) &= -\frac{1}{2m}\bar{v}_r(p)\left[(p-q)^\mu + i\sigma^{\mu\nu}(p+q)_\nu\right]v_{r'}(q)\end{aligned}$$

and the orthogonality relations of the spinors $u^*_r(p)$ and $v_{r'}(q)$

We see that the expectation value of the amplitude of the Dirac wavepacket contains components that oscillate in time with the amplitude $\frac{1}{E}$.