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# Advanced Quantum Theory WS 2012 / 2013

## Exercise Sheet 5

(To be handed in on 16. November and discussed on 19. and 20. November)

### 5.1 Charged Particle in a Magnetic Field (2 + 5 + 1 Points)

It has been shown in the lecture that in the nonrelativistic limit, the Dirac equation reproduces the *Pauli equation* for a charged particle with spin $\frac{1}{2}$ with a correct value for the Landé factor $g = 2$. Its numerical value can be directly read off from the prefactor of the term which couples the spin (vector of Pauli matrices) to the magnetic field (refer to lecture notes). Unlike the case of Dirac equation, there is no obvious way of taking into account the spin degree of freedom in the Schrödinger equation, unless it is inserted in an *ad hoc* manner. However the coupling between the *orbital* angular momentum and the magnetic field can be recovered, and this is what we compute below.
We consider a particle of mass $m$ and charge $e$ in an electromagnetic field. The classical Hamiltonian of such a particle is given by

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + e\Phi(\vec{x},t), \tag{1}$$

where $\Phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ are the scalar and vector potentials, respectively. The electric and magnetic fields can be represented in the usual way:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{2}$$

a) Show that for a homogeneous magnetic field $\vec{B}$ the vector potential $\vec{A}(\vec{x},t)$ can be written as follows:

$$\vec{A} = -\frac{1}{2}\left(\vec{x} \times \vec{B}\right) \tag{3}$$

b) By appealing to the correspondence principle, write down the Hamiltonian (1) in operator form and hence the Schrödinger equation. Show that if one imposes the *Coulomb gauge* $\vec{\nabla} \cdot \vec{A} = 0$ and neglects a term proportional to $\vec{A}^2$, one obtains the Schrödinger equation with an additional term that describes the coupling to the magnetic field:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e}{2mc}\vec{L}\cdot\vec{B} + e\Phi\right)\psi(\vec{x},t) \tag{4}$$

c) Comment on the prefactor of the coupling term in Eq. (4).

### 5.2 Discrete Symmetries of the Dirac Equation (6 + 6 + 6 Points)

In this exercise we consider the following symmetry transformations of the Dirac equation: charge conjugation, time reversal and parity transformation. We start from the canonical form of the Dirac equation which is minimally coupled to an electromagnetic field,

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left(c\vec{\alpha}\cdot\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}(\vec{r},t)\right) + \beta mc^2 + e\Phi\right)\Psi(\vec{r},t). \tag{5}$$

The same logical reasoning should be followed in each of the following exercises: first, using physical arguments, check that you obtain the respective equations (Eqs. (6), (9) and (12)) after performing each of the symmetry transformations listed above. Then, postulate the existence of a symmetry operator that will revert each of those equations back to the Dirac form, Eq. (5). Hence, find a transformation rule that this symmetry operator must obey with regards to the matrices $\vec{\alpha}$ and $\beta$. Finally, check that the explicit form of the matrices given are respectively fulfilled by the transformation rules.

a) Using physical arguments, show that the charge-conjugated Dirac equation is given by:

$$i\hbar\frac{\partial}{\partial t}\Psi_C(\vec{r},t) = \left(c\vec{\alpha}\cdot\left(\frac{\hbar}{i}\vec{\nabla} + e\vec{A}(\vec{r},t)\right) + \beta mc^2 - e\Phi\right)\Psi_C(\vec{r},t), \tag{6}$$

where $\Psi_C(\vec{r},t)$ is the charge-conjugated Dirac wavefunction. Hence, show that the charge conjugation operator fulfills the relation

$$\Psi_C(\vec{r},t) = C\Psi(\vec{r},t) = \widetilde{C}\Psi^*(\vec{r},t), \tag{7}$$

where

$$\widetilde{C} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

*Hint: Try taking the complex conjugate of Eq.* (5) *and multiplying by* $-1$. *Now multiply the thus-obtained equation from the left with a postulated operator* $\widetilde{C}$ *and use the fac that* $\widetilde{C}^{-1}\widetilde{C} = \mathbb{1}$ *on the right-hand side. What is the necessary condition for* $\vec{\alpha}$ *and* $\beta$ *in this equation such that you recover the charge conjugated Dirac equation* (6)*? Similar procedures will be repeated in all the other exercises!*

b) We do the same for the case of time reversal. For this case, show that the time reversed *Dirac-like* (hence no T-subscript on $\Psi(\vec{x},t)$) equation has the form

$$-i\hbar\frac{\partial}{\partial t'}\Psi(\vec{r},t') = \left(c\vec{\alpha}\cdot\left(\frac{\hbar}{i}\vec{\nabla} + e\vec{A}(\vec{r},t')\right) + \beta mc^2 + e\Phi\right)\Psi(\vec{r},t') \tag{9}$$

where $t \to -t = t'$. Performing similar reasonings as in (a), show that the time reversal operator can be represented as

$$\Psi_T(\vec{r},t') = T\Psi(\vec{r},t') = \widetilde{T}\Psi^*(\vec{r},t), \tag{10}$$

where

$$\widetilde{T} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{11}$$

c) Finally, we do the same for parity transformation. Show that the obtained parity-transformed Dirac equation is given by

$$i\hbar\frac{\partial}{\partial t}\Psi_P(\vec{r}\,',t) = \left(c\vec{\alpha}\cdot\left(-\frac{\hbar}{i}\vec{\nabla}\,' + e\vec{A}(\vec{r}\,',t)\right) + \beta mc^2 - e\Phi(\vec{r}\,')\right)\Psi_P(\vec{r}\,',t). \tag{12}$$

Again performing the same reasoning as exercises (a) and (b) above show that the parity transformation operator is defined as

$$\Psi_P(\vec{r}\,',t) = \widetilde{P}\Psi(\vec{r},t), \tag{13}$$

where

$$\widetilde{P} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{14}$$