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Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
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Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 4

(To be handed in on 9. November and discussed on 12. and 13. November)

<u>4.1 Transformation Properties of Ψ</u> $(2+5+2+2$ Points)

It has been shown in the lecture that a spinor Ψ transforms under a general Lorentz transformation L in the following way:

$$\Psi'(\bar{x}') = S(L)\Psi(\bar{x}) = S(L)\Psi(L^{-1}\bar{x}') \tag{1}$$

where $S(L)$ is a 4×4 matrix which operates on the 4 components of the spinor $\Psi(\bar{x})$. The principle of invariance of physical laws for all inertial systems ensures that the inverse $S^{-1}(L)$ exists.

a) Use the covariance property of the Dirac equation to show that the following identity for the transformation matrix $S(L)$ holds:

$$S(L)^{-1}\gamma^\nu S(L) = L^\nu{}_\mu \gamma^\mu \tag{2}$$

assuming $\gamma'^\nu = \gamma^\nu$, where γ'^ν is the gamma matrix in the primed reference frame.

b) Using (2), derive the relation

$$S^\dagger \gamma^0 = b\gamma^0 S^{-1} \tag{3}$$

where b is a yet undetermined factor.

c) Use the expression proven in part (b) to show that the factor b must be real, and in addition that $b = \pm 1$.
(*Hint: Consider the determinant of Eq.* (3) *for the second part, and use the normalization* $\det S = +1$)

d) Finally, show that

$$b = \begin{cases} +1 & \text{for } L^0{}_0 \geq 1 \\ -1 & \text{for } L^0{}_0 \leq -1. \end{cases} \tag{4}$$

holds.
(*Hint: Take into account the fact that the eigenvalues of* $S^\dagger S$ *are positive definite, and consider its trace. Use the facts that* $S^\dagger S = S^\dagger \gamma^0 \gamma^0 S$ *and Tr* $\alpha^i = 0$)

<u>4.2 Rotation of Spinors</u> $(2+2+5$ Points)

In this exercise we will be concerned with further transformation properties of Dirac spinors, in particular its behaviour under rotations. We consider again Eq. (1), where one can represent an infinitesimal Lorentz transformation in Minkowski space and Dirac spinor space as,

$$L^\nu{}_\mu = \delta^\nu{}_\mu + \Delta\omega^\nu{}_\mu \qquad \text{and} \qquad S(L) = 1 + \frac{1}{8}\Delta\omega^{\mu\nu}[\gamma_\mu, \gamma_\nu] \tag{5}$$

respectively.

a) Show that for infinitesimal rotations in three spatial dimensions around the rotation axis $\Delta\vec{\phi}$ with the rotation angle $|\Delta\vec{\phi}|$, one can express the parameters $\Delta\omega^{ij}$ as follows:

$$\Delta\omega^{ij} = -\varepsilon^{ijk}\Delta\phi^k \tag{6}$$

b) Show that we can write down the following representation for $\sigma_{ij} = \frac{i}{2}\left[\gamma_i, \gamma_j\right]$, $i,j,k = 1,2,3,\ i \neq j$:

$$\sigma_{ij} = \varepsilon^{ijk}\Sigma^k \tag{7}$$

with

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{8}$$

where σ^k is the kth Pauli matrix and ε^{ijk} the totally antisymmetric tensor (the Levi-Civita tensor) of third rank.

(*Hint: For this question, and in Exercise 4.3(a) below, use the following identity of the Pauli matrices:*)

$$\sigma_i\sigma_j = \delta_{ij} + i\sum_k \varepsilon_{ijk}\sigma_k$$

c) Show with the help of relation (1) that a spinor transforms under infinitesimal spatial rotations in three dimensions in the following way,

$$\Psi'(\bar{x}) = \left(1 + i\Delta\phi^k J^k\right)\Psi(\bar{x}) \tag{9}$$

with

$$J^k = \left[\varepsilon^{kij}x^i\left(-i\frac{\partial}{\partial x^j}\right) + \frac{1}{2}\Sigma^k\right]. \tag{10}$$

Hence, the generators of the rotation group in position and spinor space are J^k. In addition, J^k are also the components of the total momentum operator (orbital angular momentum and spin).

(*Hint: Consider the infinitesimal Lorentz transformation of the spinor Ψ and the 4-vector $\bar{x}$, and Taylor expand the spinor in the small parameter $\Delta\omega^{ij}$. Be careful of indices!*)

<u>4.3 Free Dirac Particle</u> (2 + 5 Points)

It is known from the lecture that the Dirac density ρ and the Dirac current density $\vec{j}$ are given by the following expressions:

$$\rho = \psi^\dagger\psi \qquad \vec{j} = \psi^\dagger(c\vec{\alpha})\psi \tag{11}$$

We consider a free Dirac particle with energy ε and an arbitrary momentum $\vec{p}$. We can write the wavefunction of such a particle as

$$\psi = u(\varepsilon, \vec{p})\exp\left(\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - \varepsilon t)\right). \tag{12}$$

a) Show that the solution of the Dirac equation for a particle and an antiparticle is determined by

$$u(\varepsilon = E, \vec{p}) = N\begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\varphi \end{pmatrix} \quad \text{and} \quad u(\varepsilon = -E, \vec{p}) = N\begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{-E+mc^2}\varphi \end{pmatrix}, \tag{13}$$

respectively, where $E = +\sqrt{\vec{p}^2c^2 + m^2c^4} \geq mc^2$. Use the form of the Dirac equation $i\hbar\frac{\partial}{\partial t}\psi = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi$. The 2−Spinor φ is a solution of the equation $(\vec{\sigma}\cdot\vec{p})\,\varphi = \pm|\vec{p}|\,\varphi$. Show that with the normalization factor $N^2 = \left(\varepsilon + mc^2\right)/2\varepsilon$ the 4-spinor u and the 2-spinor φ have the same norm, i.e., $u^\dagger u = \varphi^\dagger\varphi$.

(*Hint: Use the following identity of the Pauli matrices:*

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b}) + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

b) Use the definition of the Dirac density ρ and the Dirac current density $\vec{j}$ for the solution of the free Dirac equation in (11) and show that for a *particle*, the Dirac current density $\vec{j}$ is given as follows:

$$\vec{j} = \rho\vec{v} \tag{14}$$

Note that for the case of an *antiparticle*, for a fixed direction of motion (momentum), we find the opposite sign of the current density due to the factor of ε in the denominator.

(*Hint: The velocity $\vec{v}$ is given by $\vec{v} = \partial\varepsilon/\partial\vec{p} = c^2\vec{p}/\varepsilon$*)