Universität Bonn 6. October 2012
Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
www.uni-bonn.de/kroha/teaching/lectures-and-seminars/ws2012/aqt

Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 3

(To be handed in on 2. November and discussed on 5. and 6. October)

3.1 Manipulation of Dirac Matrices (2 + 2 + 2 Points)

It was shown in the lecture that the Dirac equation can be written in the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x^\mu) = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\Psi(x^\mu), \tag{1}$$

where $\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ are *Dirac matrices* which are composed of the Pauli matrices σ^i and the 2×2 unity matrix $\mathbb{1}$. The solutions Ψ are 4-component *Dirac spinors.*

a) Verify the relations $[\alpha^i, \alpha^j]_+ = 2\delta^{ij}\mathbb{1}$ and $[\alpha^i, \beta]_+ = 0$, with $a, b = x, y, z$. Show that in general these anticommutation relations can only be satisfied by matrices $M_{n\times n}$ with even n.

b) Rewrite the Dirac equation (1) using the gamma matrices $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. Show the relations $(\gamma^0)^\dagger = \gamma^0, (\gamma^i)^\dagger = -\gamma^i$, with $i = 1, 2, 3$ and $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\mathbb{1}$, where $(\gamma^0)^2 = \mathbb{1}$ and $(\gamma^i)^2 = -\mathbb{1}$ using general commutation properties of the α^i and β matrices.

c) Prove that the anticommutation relation $[\tilde{\gamma_\mu}, \tilde{\gamma_\nu}]_+ = 2g^{\mu\nu}\mathbb{1}$ is valid for the rotated gamma matrices $\tilde{\gamma_\mu} = A\gamma_\mu A^{-1}$. Show that the matrices A are unique for given $\tilde{\gamma_\mu}$ and γ_μ up to a constant factor.

3.2 Klein-Gordon equation in Schrödinger form (2 + 1 + 2 + 1 + 3 Points)

For some calculations it is useful to write the Klein-Gordon equation in a 2-component form. We define the following wave functions:

$$\chi_1 = \frac{1}{2}\left[\phi + \frac{i}{m}(\partial^0 + ieA^0)\phi\right], \qquad \chi_2 = \frac{1}{2}\left[\phi - \frac{i}{m}(\partial^0 + ieA^0)\phi\right] \tag{2}$$

where ϕ is a solution to the Klein-Gordon equation. We easily see that $\phi = \chi_1 + \chi_2$ and $\chi_1 - \chi_2 = \frac{i}{m}(\partial^0 + ieA^0)\phi$. The charge density (with $\hbar = c = 1$) is then given by

$$\rho = \frac{ie}{2m}[\phi^*(\partial^0 + ieA^0)\phi - \phi(\partial^0 + ieA^0)\phi^*] = e(|\chi_1|^2 - |\chi_2|^2), \tag{3}$$

which is rather simple and somewhat similar to the non-relativistic case.

a) Show that χ_1 and χ_2 obey the following coupled equations:

$$(i\partial^0 - eA^0)\chi_1 = \frac{1}{2m}(-i\vec{\nabla} - e\vec{A})^2(\chi_1 + \chi_2) + m\chi_1 \tag{4}$$

$$(i\partial^0 - eA^0)\chi_2 = -\frac{1}{2m}(-i\vec{\nabla} - e\vec{A})^2(\chi_1 + \chi_2) - m\chi_2 \tag{5}$$

b) We define the 2-component spinor

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.$$

Show that (4) and (5) together with the Pauli-matrices σ_2 and σ_3 can be combined to obtain the spinor equation

$$(i\partial^0 - eA^0)\chi = [\frac{1}{2m}(-i\vec{\nabla} - e\vec{A})^2(\sigma_3 + i\sigma_2) + m\sigma_3]\chi \tag{6}$$

Equation (6) is a first order Schrödinger equation. Also, in this notation the charge density can be written as

$$\rho = e(|\chi_1|^2 - |\chi_2|^2) = e\chi^\dagger \sigma_3 \chi$$

The normalization condition becomes:

$$\langle \chi | \chi \rangle = \int d^3x \chi^\dagger(x)\sigma_3\chi(x) = \pm 1$$

It will be shown later that the sign is determined by whether we start with particle (+) or anti-particles (−). Consider the free particle solutions

$$\chi_1 = \frac{1}{2}\left[\phi + \frac{i}{m}\partial^0\phi\right], \qquad \chi_2 = \frac{1}{2}\left[\phi - \frac{i}{m}\partial^0\phi\right]$$

A positive-energy plane-wave solution, which is normalized to unit density, is given by

$$\phi(x) = \sqrt{\frac{m}{E}} e^{-i(Et - \vec{p}\cdot\vec{x})} = \sqrt{\frac{m}{E}} e^{-ip\cdot x}.$$

c) Show that one can write

$$\chi = \chi^{(+)}(\vec{p})e^{-ip\cdot x},$$

where $\chi^{(+)}(\vec{p}) \equiv \frac{1}{2\sqrt{mE}}\binom{m+E}{m-E}$.

The corresponding negative-energy solution can be written as $\chi^{(-)}(\vec{p}) \equiv \frac{1}{2\sqrt{mE}}\binom{m-E}{m+E}$

d) Show that the spinor components fulfill the following orthogonality conditions:

$$\begin{aligned} \langle \chi^{(+)}(\vec{p})|\chi^{(+)}(\vec{p})\rangle &\equiv \chi^{(+)^\dagger}(\vec{p})\sigma_3\chi^{(+)}(\vec{p}) = 1 \\ \langle \chi^{(-)}(\vec{p})|\chi^{(-)}(\vec{p})\rangle &= -1 \\ \langle \chi^{(\pm)}(\vec{p})|\chi^{(\mp)}(\vec{p})\rangle &= 0 \end{aligned}$$

By completeness, any wavepacket can be expanded in terms of a linear combination of positive- and negative-energy solutions:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}[\alpha_{\vec{p}}^{(+)}(t)\chi^{(+)}(\vec{p})e^{i\vec{p}\cdot\vec{x}} + \alpha_{\vec{p}}^{(-)}(t)\chi^{(-)}(\vec{p})e^{-i\vec{p}\cdot\vec{x}}]$$

e) Show that if the wavefunction is normalized to unity we have:

$$\begin{aligned} 1 \equiv \langle \phi|\phi\rangle &= \int d^3x \phi^\dagger(\vec{x}, t)\sigma_3\phi(\vec{x}, t) \\ &= \int \frac{d^3p}{(2\pi)^3}\left[|\alpha_{\vec{p}}^{(+)}(t)|^2 - |\alpha_{\vec{p}}^{(-)}(t)|^2\right] \end{aligned}$$