Universität Bonn 23. Oktober 2012
Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
www.uni-bonn.de/kroha/teaching/lectures-and-seminars/ws2012/aqt

physics606 Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 2

(To be handed in on 26. October and discussed on 29. and 30. October)

2.1 Klein-Paradox for spin-0 particles

In analogy to non-relativistic quantum mechanics, this problem examines the issue of a Klein-Gordon particle incident on a step function potential. The coupling of the electromagnetic field is taken into account via *minimal coupling*, as in the Hamiltonian of the Schrödinger equation:

$$i\hbar\partial^\mu \to i\hbar\partial^\mu - \frac{q}{c}A^\mu \quad with \quad A_\mu = (\Phi(x), 0, 0, 0) \tag{1}$$

The electrostatic potential $V(x) = q\Phi(x)$ is defined by: (*Note that here and in the following, no covariant notation will be used and only the x-component of the according vectors is considered.*)

$$V(x) = \begin{cases} 0 & x < 0 \quad \text{(I)} \\ V_0 & x \geq 0 \quad \text{(II)} \end{cases} \tag{2}$$

(a) Write down the Klein-Gordon equation in regions (I) and (II).

(b) Using the following ansatz:

$$\Psi_<(x) = e^{-i\omega t}(e^{ikx} + Re^{-ikx}), \qquad x < 0 \tag{3}$$

$$\Psi_>(x) = Te^{-i\omega t}e^{ik'x}, \qquad x \geq 0 \tag{4}$$

show that the wavefunctions in (3) and (4) are solutions of the Klein-Gordon equation in the respective regions $x \geq 0$ and $x < 0$ (where $p = \hbar k$, $p' = \hbar k'$) with momenta given by

$$p = \sqrt{E^2/c^2 - m^2c^2}, \qquad p' = \pm\sqrt{(E - V_0)^2/c^2 - m^2c^2} \tag{5}$$

with $E = \hbar\omega$.

(c) Show that in this case there exist 3 different regimes for the potential V_0 (*weak*, *intermediate*, *strong potential*), wherein there are qualitatively distinct solutions for p'. Calculate those and compare to the non-relativistic case.

(d) Calculate R and T by exploiting the necessary boundary conditions (the wavefunction and its derivative are continuous at $x = 0$).

(e) The reflection coefficent ρ and the transmission coefficient τ are given by the by the ratio of the currents in the respective parts (I) and (II) to the incident current j_{in} in the following manner:

$$\rho = 1 - j_</j_{in} \quad \tau = j_>/j_{in}$$

Calculate both coefficients in the 3 potential regimes and discuss the physical result.
(Note: The current in x-direction for a wave Ψ is given by $j_x = \frac{\hbar}{2mi}(\Psi^*\partial_x\Psi - \Psi\partial_x\Psi^*)$.)

2.2 Lorentz transformation of an electromagnetic potential

We want to investigate the behavior of an electromagnetic potential under a transformation into another inertial frame. For simplicity we restrict ourselves to one spatial dimension. Let $A^\mu(x,t) = \begin{pmatrix} \phi(x,t) \\ a(x,t) \\ 0 \\ 0 \end{pmatrix}$ be the 4-potential in reference frame S.

(a) Transform $A^\mu(x,t)$ into a reference frame S' moving with velocity $-v$ in the x-direction.

(b) Now assume $a(x,t) = 0$. What is the transformed $A'^\mu(x',t')$? Is there a magnetic field $\vec{B}$ in this case? Why or why not?
(*Partial answer: In this particular case, there is no magnetic field. Contrast this with the case of a point charge in two inertial frames S and S' in which a magnetic field is observed in the moving frame. Think about the charge distribution for the particular case of this vector potential.*)

(c) Transform the potential $a(x,t) = 0$, $\phi(x,t) = \phi(x) = \frac{1}{e^{-x/b}+1}$ and sketch the result for $t' = 0$. Compare to the initial, not transformed system. What happens, if one chooses $\phi(x) = \delta(x)$ instead?

2.3: Gaussian Wavepacket

We consider a Gaussian wavepacket in one spatial dimension (*Again, note that here and in the following, <u>no</u> covariant notation will be used and only the x-component of the according vectors is considered.*)

$$\varphi(t,x) = \int \mathrm{d}k\; A(k) e^{i(kx-\omega(k)t)}, \qquad A(k) \propto \exp\left[-\frac{(k-k_0)^2}{2\sigma^2} \right] \tag{6}$$

(a) Given the following dispersion relation $\omega(k) = c\sqrt{k^2 + (\frac{mc}{\hbar})^2}$, show that the Gaussian wavepacket is a solution of the Klein-Gordon equation.

(b) At time $t = 0$ the wavepacket has the following form in coordinate representation:

$$\varphi(0,x) \propto \exp\left[-\frac{x^2}{2\sigma_x^2} \right] e^{i(k_0 x)} \tag{7}$$

with $\sigma_x(0) \equiv \sigma_x \equiv 1/\sigma$. Show that the width of the wavepacket in coordinate representation at time t can be written as

$$\sigma_x(t) = \sqrt{\sigma_x^2 + \left(\frac{\alpha t}{\sigma_x}\right)^2}, \qquad \alpha = \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}|_{k=k_0} \tag{8}$$

Use the expansion of the dispersion $\omega(k)$ up to 2nd order

$$\omega(k) \approx \omega(k_0) + (k-k_0)\frac{\mathrm{d}\omega}{\mathrm{d}k}|_{k=k_0} + (k-k_0)^2 \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}|_{k=k_0} \tag{9}$$

where $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ is the *group velocity*.
Hint: Use the fact that $\int_{-\infty}^{+\infty} \mathrm{d}x\; e^{-ax^2} = \sqrt{\frac{\pi}{a}}$, Re $a > 0$. In addition, as much as possible, try to bring your final expression into a form which resembles a Gaussian wavepacket. You can read off the expression for $\sigma_x^2(t)$ by calculating the modulus squared $|\varphi(t,x)|^2$.

(c) Transform the wave packet $\varphi(t,x)$ (6), with the dispersion relation $\omega(k)$ given in problem (a), to a reference S' frame which is moving at speed $-v$ along the x-direction. Draw a sketch of the modulus of the wave packet in the original frame for times $t = 0$ and for a time $t > 0$ as a function of x. Now draw a sketch of the wavepacket as seen in the moving frame S' for time $t = 0$ and for time $t > 0$, i.e. $\varphi'(0,x)$ and $\varphi'(t > 0, x)$. Does the shape of the wavepacket change in going over from the original frame to the moving frame? Why or why not does this happen?
Hint: Do you obtain something meaningful? How should one do the transformation correctly?