Universität Bonn 11.01.2013
Physikalisches Institut
Prof. Dr. J. Kroha, Zhong Yuan Lai, Ammar Nejati
www.uni-bonn.de/kroha/teaching/lectures-and-seminars/ws2012/aqt

Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 13

(To be handed in on 18. January and discussed on 21. and 22. January)

13.1 Bound States of δ-Potentials in d Dimensions (2 + 9 + 9 Points)

We consider a particle with dispersion $\epsilon_p = \frac{p^2}{2m}$ and bound in a potential $V(\vec{x})$:

$$H = \frac{p^2}{2m} + V(\vec{x}). \tag{1}$$

We would like to determine the conditions under which the potential $V(\vec{x}) = V_0\delta^d(\vec{x})$ in $d = 1, 2, 3$ possesses a bound state.

a) Using Dyson's equation in position space, show that the full, local Green's function at position $\vec{x} = \vec{x}' = 0$ can be written as

$$G^r(\vec{x} = \vec{x}' = 0, E) = \frac{1}{G_0^r(0, E)^{-1} - V_0}. \tag{2}$$

b) Calculate the free, local retarded Green's function in $d = 1, 2, 3$ dimensions,

$$G_0^r(\vec{x} = \vec{x}' = 0, E) = \int_{|\vec{p}|\leq p_0} \frac{d^d p}{(2\pi\hbar)^d} \frac{1}{E - p^2/2m + i\eta}. \tag{3}$$

Sketch the real and imaginary parts of $G_0^r(0, E)$ for all dimensions.
Hint: As the integral for $d > 1$ diverges, upper cutoffs for the momentum, p_0 and the corresponding energy, $D = p_0^2/2m$ respectively must be introduced. First calculate the imaginary part of (5). Then calculate the real part, using the pricipal value theorem.

c) Show, with the help of (2), that a bound state always exists for $d = 1, 2$ for $V_0 < 0$. For the case of $d = 3$, a bound state could only exist when the condition $V_0 \leq V_{crit} < 0$ is satisfied, where V_{crit} is a critical, attractive potential. Determine in all cases the the binding energy, i.e., the energy of the bound state and V_{crit} (use for this purpose the sketches from exercise b)) and discuss their dependence on V_0.

13.2 The Born Approximation (3 + 3 + 3 + 3 Points)

The differential cross section for elastic potential scattering is given in the Born approximation by the expression

$$\frac{d\sigma}{d\Omega} = \frac{2\pi m^2}{\hbar^4}\left| \int d^3r V(\vec{r}) e^{i\vec{r}\vec{q}} \right|^2 \tag{4}$$

where $\vec{q} = \vec{k_i} - \vec{k_f}$ und $k = |\vec{k_i}| = |\vec{k_f}|$, where $k = |\vec{k_i}|$ and $|\vec{k_f}|$ are the incoming and outgoing momenta respectively.

a) Show that the absolute value of the momentum transfer is given by $|\vec{q}| = 2k \sin\frac{\theta}{2}$, where θ is the scattering angle.

b) For a radially symmetric potential one can carry out the angular integration separately. Show that in this case we obtain for (4) the expression:

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\hbar^4}\frac{1}{|\vec{q}^2|}\left|\int drrV(r)\sin(|\vec{q}|r)\right|^2. \tag{5}$$

c) We consider now the Yukawa potential $V(r) = \frac{\kappa}{r}\exp(-r/r_0)$. Show that the differential cross section for this case is given by the expression

$$\frac{d\sigma}{d\Omega} = \left(\frac{\kappa}{4E\sin^2\frac{\theta}{2} + \frac{\hbar^2}{2mr_0^2}}\right)^2 \tag{6}$$

d) Show that for a bell-shaped potential $V(r) = V_0\exp(-\frac{r^2}{2r_0^2})$ that the differential cross section is given by the expression

$$\frac{d\sigma}{d\Omega} = \frac{2\pi m^2 r_0^6 V_0^2}{\hbar^4} e^{-4k^2r_0^2\sin^2\frac{\theta}{2}} \tag{7}$$

How does this scattering cross section change for large scattering angles?