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Physikalisches Institut
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Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 12

(To be handed in on 18. January and discussed on 21. and 22. January)

12.1 The Resonant Level Model (2 + 8 + 2 + 6 Points)

We consider a model where a single, discrete level ε_d is coupled via a hybridization V to a conduction electron sea with dispersion $\varepsilon_{\mathbf{k}}$,

$$\begin{aligned} H &= H_0 + H_d + H_V \\ &= \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \varepsilon_d \sum_{\sigma} d^\dagger_\sigma d_\sigma + V \sum_{\mathbf{k}\sigma} (d^\dagger_\sigma c_{\mathbf{k}\sigma} + c^\dagger_{\mathbf{k}\sigma} d_\sigma). \end{aligned} \tag{1}$$

with fermionic operators $c^\dagger_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}$ and $d^\dagger_\sigma, d_\sigma$ for the conduction electrons and the electrons in the discrete level, respectively.

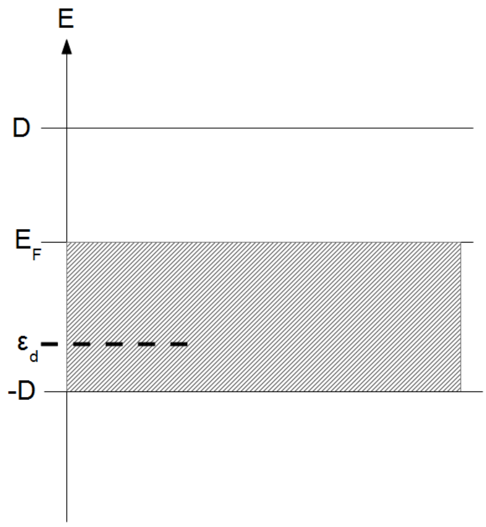

Figure 1: Coupling of an impurity level ε_d to a continuum of states. E_F is the Fermi level, while the range $[-D, D]$ denotes the bandwidth.

a) Derive the equation of motion for the free *retarded* conduction electron Green's function

$$G^R_{c\sigma}(\mathbf{k}, \mathbf{k}', t - t') = -i\theta(t - t')\langle\{c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}'\sigma}(t')\}\rangle, \tag{2}$$

i.e., according to H_0

b) We now take the full Hamiltonian H in (1) into account. Similarly to a), derive the equations of motion for the conduction electron Green's function $G^R_{c\sigma}(\mathbf{k}, \mathbf{k}', t - t')$ and the *impurity* Green's function

$$G^R_{d\sigma}(t - t') = -i\theta(t - t')\langle\{d_\sigma(t), d^\dagger_\sigma(t')\}\rangle \tag{3}$$

and obtain the Green's functions $G^R_{c\sigma}(\mathbf{k}, \mathbf{k}', \omega)$ and $G^R_{d\sigma}(\omega)$ in frequency space.

c) As on the previous exercise sheet, bring $G^R_{d\sigma}(\omega)$ into a form where the self-energy $\Sigma(\omega)$ can be written down explicitly, and show that the *retarded* self-energy is given by

$$\Sigma^R(\omega) = \sum_{\mathbf{k}} \frac{|V|^2}{\omega - \varepsilon_{\mathbf{k}} + i\eta} \tag{4}$$

where η is an infinitesimally small constant.
Remark: When you write down the equation of motion for $G^R_{d\sigma}(t-t')$, another type of Green's function

$$G^R_{cd\sigma}(\mathbf{k}, t-t') = -i\theta(t-t')\langle\{c_{\mathbf{k}\sigma}(t), d^\dagger_\sigma(t')\}\rangle, \tag{5}$$

will be generated. In order to have an equation that only contains $G^R_{d\sigma}(\omega)$, write down a second equation of motion for $G^R_{cd\sigma}(\mathbf{k}, \omega)$ and solve the resulting set of two coupled equations for $G^R_{d\sigma}(\omega)$. The same procedure applies for the case of the conduction electron Green's function, the only difference being that in this case $G^R_{dc\sigma}(\mathbf{k}', t-t') = -i\theta(t-t')\langle\{c_{d\sigma}(t), c^\dagger_{\mathbf{k}'\sigma}(t')\}\rangle$ is generated instead.

Hint: First derive the equation of motion for $G^R_{d\sigma}(\omega)$. For the case of $G^R_{c\sigma}(\mathbf{k}, \mathbf{k}', \omega)$, use the expression for $G^R_{cd\sigma}(\mathbf{k}, \omega)$ from the previous derivation. Also use the fact that $d^\dagger_\sigma$ and d_σ obey the fermionic anticommutation relations, and that at equal times $\left\{d_\sigma, c^\dagger_{\mathbf{k}\sigma}\right\} = \left\{d^\dagger_\sigma, c^\dagger_{\mathbf{k}\sigma}\right\} = 0$

d) The momentum summation of a function of $\varepsilon_{\mathbf{k}}$ only, $F(\varepsilon_{\mathbf{k}})$, can be written as

$$\sum_{\mathbf{k}} F(\varepsilon_{\mathbf{k}}) = \int_{-\infty}^{+\infty} dE\ N(E) F(E) \tag{6}$$

where

$$N(E) = \sum_{\substack{\mathbf{k} \\ \varepsilon_{\mathbf{k}=E}}} \mathbf{1} \tag{7}$$

is called the density of states at energy E. For this problem we assume

$$N(E) = \begin{cases} \frac{1}{2D} & \text{for } -D < E < D, \\ 0 & \text{for } \quad D < |E|. \end{cases} \tag{8}$$

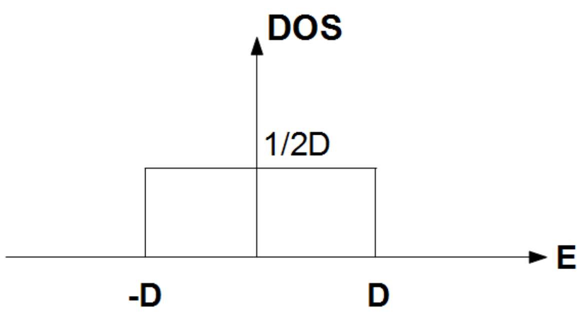

Figure 2: Density of states

Hence split the integral in (6) for the self-energy into a real and imaginary part, and for $\omega \in [-D, D]$ evaluate the real part of the integral from $-D$ to D. Thus show, using the expression from the lecture, that the spectral function of $G^R_{d\sigma}(\omega)$ shows a resonance structure with a constant width and write down the explicit expression of the width.