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Physikalisches Institut
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Advanced Quantum Theory WS 2012 / 2013

Exercise Sheet 10

(To be handed in on 21. December and discussed on 7. and 8. January)

10.1 Physical quantities from the time-ordered Green's function (2 + 3 + 3 Points)

In this exercise we will see how to derive expectation values of physical quantities of a many-body system from the Green's function, in this case the particle density and current. The time-ordered Green's function is defined as,

$$G_{\sigma\sigma'}(x,x') = -i\langle \mathcal{T}\psi_\sigma(x)\psi^\dagger_{\sigma'}(x')\rangle, \qquad \textit{with}\ \ x = (t,\vec{x})\ \text{etc..} \tag{1}$$

a) Knowing the expressions for the particle density and particle current of a single-particle wave function $\Psi(\vec{x})$, write down the particle density and current density operators of 2nd quantization in position representation.

b) Show that the expectation value of the *particle density* of a many-body system can be obtained from (1) by taking the limits

$$n_\sigma(x) = \pm i \lim_{\substack{\vec{x}'\to\vec{x}\\ t'\to t+0}} G_{\sigma\sigma}(x,x') \tag{2}$$

c) Show that the *particle current density* is obtained from the Green's function as follows:

$$\langle j_\sigma\rangle = \pm\frac{\hbar}{2m} \lim_{\substack{\vec{x}'\to\vec{x}\\ t'\to t+0}} (\vec{\nabla}_{\vec{x}} - \vec{\nabla}_{\vec{x}'}) G_{\sigma\sigma}(x,x') \tag{3}$$

In both cases the plus and minus signs refer to bosons and fermions, respectively.

10.2 Coherent States (2 + 4 + 2 + 3 + 3 Points)

We consider one single-particle state $|\alpha\rangle$ and the *bosonic* creation and destruction operators $a^\dagger$, a for a particle in that state. The normalized occupation-number eigenstate in Fock space with n particles in state $|\alpha\rangle$ is denoted by $|n\rangle$, $n = 0, 1, 2, \ldots$. We want to construct eigenstates $|\Phi\rangle$ of a,

$$a|\Phi\rangle = \phi\ |\Phi\rangle \tag{4}$$

with yet unknown eigenvalue ϕ. Clearly, $|\Phi\rangle$ cannot be an occupation number eigenstate (why?). Therefore, we make the expansion ansatz,

$$|\Phi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{5}$$

with expansion coefficients c_n.

a) Show that the *normalized* occupation number eigenstates $|n\rangle$ can be written as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{6}$$

b) Combining (4), (5) and (6), and comparing the coefficients in each term of the expansion, prove the recursion relation $c_n = \frac{\phi c_{n-1}}{\sqrt{n}}$, $c_0 = 1$ and, hence,

$$|\Phi\rangle = \sum_{n=0}^{\infty} \frac{(\phi a^\dagger)^n}{n!}|0\rangle = \mathrm{e}^{\phi a^\dagger}|0\rangle \tag{7}$$

where ϕ is an arbitrary complex number.

c) Does $a^\dagger$ have an eigenstate? Why or why not? Why is it impossible to construct eigenstates of the fermionic destruction operator in this way?

d) Compute $\frac{d}{d\phi}|\Phi\rangle$ and compare with (4).

e) Compute $\langle\Phi|\Psi\rangle$ for two eigenstates $|\Phi\rangle, |\Psi\rangle$ of a. Are these states normalized and orthogonal?

Remark: The eigenstates of a are called coherent states. They play an important role in laser physics ans well as in certain field theoretical formulations.