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physics606 Advanced Quantum Theory

Exercise Sheet 1

(To be handed in on 19. October and discussed on 22. and 23. October)

1.1 Lorentz Transformations

The scalar product of two vectors $x, y \in \mathbb{R}^N$ can be defined as

$$\langle x, y\rangle = x^T \cdot g \cdot y \tag{1}$$

with a metric tensor $g \in M_{N\times N}$ We consider linear transformations $L : \mathbb{R}^N \to \mathbb{R}^N$ $\det(L) = \pm 1$ obeying:

$$\langle x, y\rangle = \langle Lx, Ly\rangle \tag{2}$$

(a) Show that for the transformations defined in (2) the relations $L^T g L = g$ hold. How should one interpret the different values of the determinant of L?

(b) *Rotations in position space.* We consider the two-dimensional Euclidean space with metric $g = \mathbb{1}$. Derive the general structure of the transformations L in this space which obeys (2) with $\det(L) = \pm 1$ and show that the components of L can be written as ${L^1}_1 = {L^2}_2 = \cos\varphi, {L^1}_2 = -{L^2}_1 = \sin\varphi$. What is the geometrical meaning of the angle φ?

(c) *Minkowski space.* Einstein realized that the light velocity c is a constant in all inertial systems. This postulate can be formulated via the quantity $s^2 = (ct)^2 - \vec{x}^2$, which is invariant (in the sense of $s = s'$, for s and s' in different inertial systems) in all inertial systems. Hence, confirm that any transformation between two inertial systems has to obey relation (2), using the *Minkowski metric* $g^{00} = 1$, $g^{aa} = -1$ with $a = 1, 2, 3$ and $g^{\mu\nu} = 0$ for $\mu \neq \nu$. Without loss of generality the x-axis is set parallel to the direction of motion in the following. Show that, analogous to (b), the transformations L now can be written as

$$L_x := \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) *Lorentz transformations.* The inertial system S' moves with the velocity $-v$ (along the x-axis) relative to S. A particle, which is at rest at the origin of S, $x^0 = ct, x^1 = 0$, has the coordinates $x^{0'} = ct', x^{1'} = vt'$ in S'. Show that due to $s^2 = s'^2$,the transformation L between S and S' consists of the components $\cosh\phi = \gamma$, $\sinh\phi = \beta\gamma$ with

$$\beta = \frac{\nu}{c} \qquad\qquad \gamma = \frac{1}{\sqrt{1 - \nu^2/c^2}} \tag{3}$$

These transformations are called *proper, orthochronous Lorentz transformations* (i.e. no reflections or translations in space-time).

1.2 Contra- and Covariant Formulation

In the contra- and covariant formulation vectors are defined by their components x^μ (contravariant) and x_μ(covariant). The corresponding vectors and their correct notation are given by

$$\overline{x} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad\qquad \underline{x} = x^T g = (ct, -x, -y, -z)$$

(in this execise g always means the Minkowski metric).

Hence the scalar product can be written as $\langle \overline{x}, \overline{y} \rangle = x_\mu y^\mu$. A linear transformation $\overline{x}' = L\overline{x}$ is defined by $x'^\mu := L^\mu{}_\nu x^\nu$.

(a) Reexpress relation (2) of Exercise 1.1 using the formulation in terms of components x^μ, x_μ etc., given above.

(b) Show for the covariant vectors: $\underline{x}' = \underline{x}L^{-1}$ with $(L^{-1})^\mu{}_\nu = g_{\nu\beta} L^\beta{}_\alpha g^{\alpha\mu} := L_\nu{}^\mu$.

(c) Prove that Lorentz transformations obey the inequality $|L_0{}^0| \geq 1$. When $|L_0{}^0| \geq 1$ and $\det(L) = 1$ then L is an element of the *proper orthochronous Lorentz group* $L_+^\uparrow$

(d) The co- and contravariant derivatives are defined as follows:

$$\partial_\mu := \frac{\partial}{\partial x^\mu} \qquad\qquad \partial^\mu := \frac{\partial}{\partial x_\mu} \tag{4}$$

(e) Why is the d'Alembert operator $\Box := \partial_\mu \partial^\mu = \partial^\mu \partial_\mu$ Lorentz invariant ?

1.3: Klein-Gordon equation and Klein-Gordon current

This execise deals with the Klein-Gordon equation

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \psi(x) = 0 \tag{5}$$

(a) By Fourier transforming (5), or using other methods, obtain the solution of the Klein-Gordon equation for arbitrary $A(k)$:

$$\psi(x) = \int \frac{d^4k}{\sqrt{2\pi}^4} \delta \left(\underline{k}\overline{k} - \left(\frac{mc}{\hbar} \right)^2 \right) A(k) e^{-i\underline{k}\overline{x}} \tag{6}$$

Note: $\overline{k}$, $\overline{x}$ are 4-vectors with components k^μ, x^μ, $(\mu = 0, 1, 2, 3)$. $\vec{k}$, $\vec{x}$ are 3-vectors with components k^a, x^a, $(a = 1, 2, 3)$. Apart from that $\underline{k}\overline{x} = k_\mu x^\mu$

Now we define a Klein-Gordon 4-current

$$j^\mu(x) = \frac{i\hbar}{2m} \left(\psi^*(x) \partial^\mu \psi(x) - \left[\partial^\mu \psi^*(x) \right] \psi(x) \right) \tag{7}$$

(b) Show that this current fulfills the continuity equation, i.e.:

$$\partial_\mu j^\mu(x) = 0 \tag{8}$$

(c) Confirm that this current is indeed a 4-vector (i.e. transforms like a Lorentz vector). *(Hint: Use the fact that $\psi'(x') = \psi(x)$, with primed and unprimed quantities in different inertial frames.)*

1.4: Classification of the solutions of the Klein-Gordon equation

Consider the solution (6) to the Klein-Gordon equation in Exercise 1.3. Define functions $\phi_\pm$:

$$\phi_+ = \frac{1}{\sqrt{2\pi}^3} \int \frac{d^3k}{2\omega_k} A(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}$$

$$\phi_- = \frac{1}{\sqrt{2\pi}^3} \int \frac{d^3k}{2\omega_k} A(\vec{k}, -\omega_k) e^{i(\vec{k}\cdot\vec{x} + \omega_k t)}$$

with the dispersion relation $\omega_k / c = \sqrt{(\vec{k})^2 + (\frac{mc}{\hbar})^2}$.

(a) Show that:

$$\phi(x) = \phi_+(x) + \phi_-(x) \tag{9}$$

Interpret the solutions ϕ_+, ϕ_-. Equation (9) shows that for a general solution of the Klein-Gordon equation, both the functions ϕ_+ and the functions ϕ_- are necessary.

(b) Since the quantity $|\psi|^2$ does not fulfill a conservation law (continuity equation), it cannot be used to normalize the wave function $\psi(x)$. Using (8), show that the 0th component of the 4-current, integrated over space,

$$\|\psi\|_c^2 := \frac{i\hbar}{2m} \int d^3x \left\{ \psi^*(x) \partial^0 \psi(x) - [\partial^0 \psi^*(x)] \psi(x) \right\}$$

is constant in time.
Therefore, $\|\psi\|_c^2$ is called the "current norm" of $\psi(x)$

(c) Show that:

$$\|\psi\|_c^2 = \|\psi_+\|_c^2 + \|\psi_-\|_c^2$$

where

$$\|\psi_\pm\|_c^2 = \pm \int \frac{d^3k}{2\omega_k} |A_\pm(\vec{k})|^2$$

Here we used $A_+(\vec{k}) := A(\vec{k}, \omega_k)$ and $A_-(\vec{k}) := A(\vec{k}, -\omega_k)$.

We thus see that for positive energy solutions ϕ_+ the normalization is strictly positive definite, whereas for negative energy solutions the norm is strictly negative definite. Give a short comment on this fact.