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Quantum Field Theory for Condensed Matter Physics
Exercise 7

(Submission date: 24.01.2020)

7.1 Diagrammatics in equilibrium (4+6=10 points)

We consider an interacting Fermi system in thermodynamic equilibrium. The action reads ($x = (\tau, \boldsymbol{r})$ etc.),

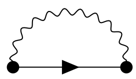
$$S\left[\Psi^\dagger, \Psi\right] = \sum_\sigma \int \mathrm{d}^4x\ \Psi^\dagger_\sigma(x)[\frac{\partial}{\partial\tau} + \frac{(-i\nabla)^2}{2m} - \mu]\Psi_\sigma(x) \tag{1}$$

$$+ \sum_{\sigma,\sigma'} \int \mathrm{d}^4x \mathrm{d}^4x'\ \Psi^\dagger_\sigma(x)\Psi^\dagger_{\sigma'}(x')V(x-x')\Psi_{\sigma'}(x')\Psi_\sigma(x), \tag{2}$$

where we assume an instantaneous, local interaction $V(x - x') = V_0\delta^4(x - x')$. In equilibrium, the situation simplifies due to time-translation invariance, so that the two-point Green function depends only on the time difference or on a single frequency, respectively.

a) The Dyson equation is the central equation of motion for the two-point Green's function. Write down the Dyson equation for a general selfenergy Σ in frequency space.

b) As an example, consider the Fock contribution to the selfenergy Σ, given diagrammatically by

 .

Label this diagram with the appropriate frequency, momentum and spin variables and evaluate it using the Matsubara technique. For an instantaneous interaction, does this diagram depend on frequency?

7.2 Diagrammatics out of equilibrium (5+5+5+5=20 points)

We now consider the field theory for the same interacting Fermi system *out of equilibrium*. The action reads again ($x = (t, \boldsymbol{r})$ etc.),

$$S\left[\Psi^\dagger, \Psi\right] = \sum_\sigma \int_C \mathrm{d}^4x\ \Psi^\dagger_\sigma(x)[-i\frac{\partial}{\partial\tau} + \frac{(-i\nabla)^2}{2m} - \mu]\Psi_\sigma(x) \tag{3}$$

$$+ \sum_{\sigma,\sigma'} \int_C \mathrm{d}^4x \mathrm{d}^4x'\ \Psi^\dagger_\sigma(x)\Psi^\dagger_{\sigma'}(x')V(x-x')\Psi_{\sigma'}(x')\Psi_\sigma(x), \tag{4}$$

where now the time integrals run over the Keldysh contour C along the real time axis.

a) The Dyson equation for the contour time-ordered two-point Green's function reads in time space for a general selfenergy $\Sigma(t_1, t_2)$

$$G(x, x') = G^{(0)}(x, x') + \int_C \mathrm{d}x_1 \mathrm{d}x_2 G^{(0)}(x, x_1)\Sigma(x_1, x_2)G(x_2, x') \tag{5}$$

Obtain the matrix elements of the Dyson equation in terms of the greater, lesser, time-ordered and anti-time-ordered Green's function and of the corresponding selfenergy. Which vertex matrices appear at the open ends of the selfenergy?

b) Transform the Dyson equation to the representation of retarded, advanced and Keldysh Green functions as defined in the lecture. Write all the matrix elements of the Dyson equation separately. Hint: In the lecture we showed

$$\begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix} = U \begin{pmatrix} G & G^< \\ G^> & \widetilde{G} \end{pmatrix} U^\dagger \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

c) Calculate the selfenergy matrix in Keldysh space in the representation (Σ^R, Σ^A, Σ^K) for the Fock diagram of problem 7.1.

d) Assume now that the system is in equilibrium, i.e., the Green's functions $G^<$ $G^>$ appearing in the selfenergy have an equilibrium Fermi-Dirac distribution. Show that your result of problem 7.2 c) agrees with the 7.1 b).

7.3 <u>The Bose-Hubbard model</u> (5+5=10 points)

We consider a spinless boson gas on a simple cubic lattice with an nearest neighbour hopping amplitude $-t$ and an repulsive contact interaction U. The particle number in the lattice is set by the chemical potential μ. The number of lattice sites is N_s. The action is given by

$$S[\overline{\phi}, \phi] = \int d\tau \left[\sum_i \overline{\phi}_i (\partial_t - \mu) \overline{\phi}_i - t \sum_{\langle i,j \rangle} \overline{\phi}_i \phi_j + \frac{U}{2} \sum_i \overline{\phi}_i \overline{\phi}_i \phi_i \phi_i \right] . \tag{6}$$

This system has two different phases. A superfluid phase and a Mott insulator phase. At zero temperature the later two are connected by a quantum phase transition.

a) Derive the Gross-Pitaevskii equation for this lattice model.

b) Calculate the condensate occupation N_0 for a static, spatially constant mean field Φ.