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Quantum Field Theory for Condensed Matter Physics
Exercise 5

(Submission date: 20.12.19)

5.1 BCS from Path integral (5+3+5+5+5+5+5+5+5+7=33+17 points)

Realistic solid state system are notoriously hard to describe, due to many different processes which are competing, like electron-electron, electron-phonon interaction or different kinds of ordered phases. In the recent decades experiments using dilute quantum gases allowed to disentangle these different contributions in idealised systems. We consider here a dilute Fermi gas interacting with a contact interaction. Experimentally this interaction can be tune by Feshbach resonances. An effective description can be archived by the s-wave scattering length. We look for cooper pairs at large negative scattering lengths.
Due to Fermi statistics only different spin state interact, so the action reads

$$S[\bar{\Psi},\Psi]=\int d^4x\sum_\sigma \bar{\Psi}_\sigma(x)(\partial_\tau-\nabla_x^2-\mu)\Psi_\sigma(x)+g\int d^4x\bar{\Psi}_\uparrow(x)\bar{\Psi}_\downarrow(x)\Psi_\downarrow(x)\Psi_\uparrow(x)\,. \tag{1}$$

a) First we introduce a general method to deal with interaction term known as the Hubbard-Stratonovich transformation. Calculate the path integral for the bosons $(\bar{\Delta},\Delta)$

$$\int\mathcal{D}(\bar{\Delta},\Delta)\exp\left\{\int d^4x\left[\frac{1}{g}\bar{\Delta}(x)\Delta(x)+\bar{\Delta}(x)\Psi_\downarrow(x)\Psi_\uparrow(x)+\Delta(x)\bar{\Psi}_\uparrow(x)\bar{\Psi}_\downarrow(x)\right]\right\}.$$

Use the result to replace the interacting term in the action.

b) The new action can be represented in matrix form by introducing Nambu spinors. They are defind by

$$\eta(x)=(\Psi_\uparrow(x),\bar{\Psi}_\downarrow(x))^T\qquad \bar{\eta}(x)=(\bar{\Psi}_\uparrow(x),\Psi_\downarrow(x))$$

Show that the action in the new presentation takes the form

$$S[\bar{\eta},\eta,\bar{\Delta},\Delta]=\int d^4x\left[\frac{-1}{g}\bar{\Delta}(x)\Delta(x)+\bar{\eta}(x)(-G_0^{-1}(x))\eta(x)\right] \tag{2}$$

$$\text{with }-G_0^{-1}(x)=\begin{pmatrix}\partial_\tau-\nabla_x^2-\mu & -\Delta(x)\\ -\bar{\Delta}(x) & \partial_\tau+\nabla_x^2+\mu\end{pmatrix}.$$

c) Transform the action into Fourier space $(r,\tau)\to(p,i\omega_n)$. Note that the interacting part in (2), the off diagonal elements of G_0^{-1}, will give rise to non-diagonal contributions. The action take the form

$$S[\bar{\eta},\eta,\bar{\Delta},\Delta]=\sum_q\left[\frac{-1}{g}\bar{\Delta}(q)\Delta(q)+\sum_k\bar{\eta}(q)(-G_0^{-1}(q,k))\eta(k)\right] \tag{3}$$

$$\text{with}\quad \eta(q)=\left(\Psi_\uparrow(q),\bar{\Psi}_\downarrow(-q)\right)^T\quad,\quad \bar{\eta}(q)=\left(\bar{\Psi}_\uparrow(q),\Psi_\downarrow(-q)\right)\quad,\quad \epsilon_k=k^2-\mu$$

$$-G_0^{-1}(q,k)=\delta_{q,k}^{(4)}\begin{pmatrix}-i\omega_k+\epsilon_k & 0\\ 0 & -i\omega_k-\epsilon_k\end{pmatrix}+\frac{1}{\sqrt{\beta V}}\begin{pmatrix}0 & -\Delta(q-k)\\ -\bar{\Delta}(k-q) & 0\end{pmatrix}$$

We first look for a saddle point approximation of the action. The first idea we will follow here is a constant mean field in space and time. Therefore we reduce the bosonic field $\bar{\Delta}_p, \Delta_p$ to the zero modes

$$\bar{\Delta}_p = \delta^{(4)}_{p,0}\sqrt{V\beta}\bar{\Delta} \qquad \Delta_p = \delta^{(4)}_{p,0}\sqrt{V\beta}\Delta\,.$$

d) Show that the Green's function is diagonal in momentum space and perform the Gaussian integral over the fermionic fields. Calculate then the determinant in Nambu space to obtain

$$Z = \exp\left\{\frac{\beta V}{g}|\Delta|^2 + \sum_k \ln\left[(i\omega_k - E_k)(-i\omega_k - E_k)\right]\right\}, \quad E_k = \sqrt{\epsilon_k^2 + |\Delta|^2}$$

Hint: Note that when introducing Nambu spinors the rearrangement of the variables introduces a sign factor in the integral measure when performing the Gaussian integral, namely

$$\int \mathcal{D}(\bar{\Psi}, \Psi) \exp\left[-\int d^4x \bar{\eta}(x) A \eta(x)\right] = \exp\left\{\left(\int d^4x \ln(-\det_\sigma(A)\right)\right\}$$

Here $\det_\sigma(A)$ denote the determinant in the 2×2 Nambu space.

In this approximation the free energy is equal to the action evaluated at the saddle point. In the continuum limit is given by

$$F_{mf} = -\frac{|\Delta|^2}{g} - \int \frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{\omega_n} \ln\left[(i\omega_k - E_k)(-i\omega_k - E_k)\right] \tag{4}$$

e) We need to determine the mean field value, which extremizes the action with respect to Δ. Note that E_k is a function of Δ. You should find that Δ needs to satisfy

$$\frac{1}{g} = \int \frac{d^3k}{(2\pi)^3}\frac{\tanh(\beta E_k/2)}{2E_k}\,.$$

This is called the gap equation.

We now improve our approximation to the free energy. For this we also consider small fluctuation in other modes.

$$\bar{\Delta}_p = \delta^{(4)}_{p,0}\sqrt{V\beta}\bar{\Delta} + \bar{\Phi}(p) \qquad \Delta_p = \delta^{(4)}_{p,0}\sqrt{V\beta}\Delta + \Phi(p)$$

The inverse Greens function can then be separated into the mean field and the fluctuating part.

$$-G_0^{-1}(q,k) = -\bar{G}^{-1}(k)\delta^{(4)}_{k,q} + F(q,k) \quad F(q,k) = \frac{1}{\sqrt{\beta V}}\begin{pmatrix} 0 & -\Phi(q-k) \\ -\bar{\Phi}(k-q) & 0 \end{pmatrix} \tag{5}$$

f) Integrate out the fermions ones more and obtain

$$Z = Z_{mf}\int \mathcal{D}[\bar{\Phi}, \Phi] \exp\left[\frac{\sqrt{V\beta}}{g}(\bar{\Delta}\Phi(0) + \Delta\bar{\Phi}(0)) + \sum_q \frac{\bar{\Phi}(q)\Phi(q)}{g} + \operatorname{Tr}\ln(1 - \bar{G}F)\right]$$

The linear terms in the fluctuations will vanish when including the linear term from the trace log. This is due to the stationarity of the action on the saddle point.

The free energy will be the sum of the saddle point part plus the fluctuations. Let us now consider how to obtain physical quantities from it.

g) Which equation determines the mean field value Δ? What determines the physical quantity $\langle\Delta\rangle$? What would be the consequence if we want to calculate the particle number from the free energy we obtained by incorporating fluctuations?

*****BONUS*****

We now expand the log up to second order in the fluctuations Φ, assuming they are small. The contribution to the partition function is

$$\int \mathcal{D}[\bar{\Phi}, \Phi] \exp\left\{\sum_q \frac{\bar{\Phi}(q)\Phi(q)}{g} - \frac{1}{2}\mathrm{Tr}[\bar{G}F\bar{G}F]\right\}. \tag{6}$$

We need to calculate the trace in Nambu space, so that we can perform the integral over the fluctuations. This will be done in serveral small steps. It might be helpful to think about the Greens functions as connecting two state $\langle p| G |q\rangle$ by a matrix element.

h) Calculate the mean field Greens function $\tilde{G}$ from its inverse.

$$\tilde{G}_{p,q} = \frac{\delta^{(4)}_{p,q}}{(i\omega_n + E_k)(i\omega_n - E_k)} \begin{pmatrix} i\omega_k + \epsilon_k & -\Delta \\ -\bar{\Delta} & i\omega_k - \epsilon_k \end{pmatrix}$$

i) Calculate the product $\tilde{G}_{p,q}F_{q,k}$ using the properties of $\tilde{G}$.

j) With the result of i) calculate $\mathrm{Tr}[\bar{G}F\bar{G}F]$.

The result can be cast in the from

$$\frac{1}{\beta V}\sum_{k,q} \frac{1}{(i\omega_k + i\omega_q)^2 - E^2_{q+k}} \frac{1}{(i\omega_k)^2 - E^2_k}$$
$$\times(\bar{\Phi}_q, \Phi_{-q}) \begin{pmatrix} (i\omega_k + i\omega_q + \epsilon_{q+k})(i\omega_k - \epsilon_k) & \Delta^2 \\ \bar{\Delta}^2 & (i\omega_k + i\omega_q - \epsilon_{q+k})(i\omega_k + \epsilon_k) \end{pmatrix} \begin{pmatrix} \Phi_q \\ \bar{\Phi}_{-q} \end{pmatrix}$$

The expression can be combined to

$$\int \mathcal{D}[\bar{\Phi}, \Phi] \exp\left[-\frac{1}{2}\sum_q (\bar{\Phi}(q), \Phi(-q)) \begin{pmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} \end{pmatrix} \mathcal{M} \begin{pmatrix} \Phi(q) \\ \bar{\Phi}(-q) \end{pmatrix}\right] \tag{7}$$

This can now be integrated as before to obtain the fluctuating contribution to the to the free energy F_{fl}.