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Quantum Field Theory for Condensed Matter Physics
Exercise 4

(Submission date: 06.12.19)

4.1 Density Correlations in a Fermi gas (8+4+4+6+8=30 points)

Due to the Pauli principle, two electrons of equal spin cannot be simultaneously at the same point in space. Because the electron wave-functions are continuous, this effect of Fermi statistics induces spatially extended, repulsive density correlations in a Fermi gas even without interactions, so-called statistical correlations. To describe this effect quantitatively, we calculate the density response function χ in three dimensions for equal spins σ.

The generating functional for this problem, including source terms, is given by

$$Z[J,\bar{J}] = \int D(\bar{\Psi},\Psi) \exp\left\{ -\int \frac{d^3k}{(2\pi)^3} \int_0^\beta d\tau \left[\bar{\Psi}_{\boldsymbol{k}}(\partial_\tau + \epsilon_{\boldsymbol{k}})\Psi_{\boldsymbol{k}} + \bar{\Psi}_{\boldsymbol{k}} J_{\boldsymbol{k}} + \bar{J}_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} \right] \right\} . \tag{1}$$

Here $\epsilon_k = \frac{k^2}{2m}$ is the dispersion of the free particle in a translational invariant system. The time ordered density correlations function $\chi(r-r',\tau)$ is then defind as

$$\chi(r-r',\tau) = -\langle \hat{T} n(r,\tau) n(r',0) \rangle \quad \text{with} \quad n(r,\tau) = \bar{\Psi}(r,\tau)\Psi(r,\tau).$$

a) Show that $\chi(r-r',\tau)$ is given in momentum representation as

$$\chi(\boldsymbol{q},\tau) = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left\langle \hat{T}\left(\bar{\Psi}_{\boldsymbol{k}}(\tau)\Psi_{\boldsymbol{k}+\boldsymbol{q}}(\tau)\bar{\Psi}_{\boldsymbol{k}'}(0)\Psi_{\boldsymbol{k}'-\boldsymbol{q}}(0) \right) \right\rangle \tag{2}$$

b) Calculate (2) by suitable derivatives of eq. (1) w.r.t. $(\bar{J}, J)$ and give the diagrammatic representation.
Hint: Useful notation $G_k(\tau) = [(\partial_\tau + \epsilon_k)^{-1}]_{k,\tau}$

c) Fourier transform the connected part of $\chi(\boldsymbol{q},\tau)$ to $\chi(\boldsymbol{q},i\omega)$. The result is given by

$$\chi(\boldsymbol{q},i\omega) = \frac{1}{\beta}\sum_{i\nu} \int \frac{d^3k}{(2\pi)^3} G_{k+q}(i\omega + i\nu) G_k(i\nu) \,.$$

d) Perform the Matsubara sum over $i\nu$. The result can be analytic continued to the retarded correlation function in real frequencies by replacing $i\omega \to \omega + i\eta$.

$$\chi^R(q,\omega) = \int \frac{d^3k}{(2\pi)^3} \frac{f(\epsilon_k) - f(\epsilon_{k+q})}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta}$$

e) Evaluate the integral at low temperature in the long wavelength and static limit $q, \omega \to 0$.

4.2 Electron-phonon interaction (12+8=20 points)

We consider here the interaction of a electron gas with an underlying ionic lattice and show that an effective electron-electron interaction emerges. The scattering of an electron on an atom, displaces the atom slightly from the equilibrium position. These excitation can be described by phonons, which are bosonic quasi particles. The relevant processes are the scattering of an electron of momentum k into the state $k - q$ by either emitting a photon with momentum q or absorbing a photon with momentum $-q$.

We use a compact notation for the momentum sum's and field indicies, namely $\Psi_{\boldsymbol{k}}(i\omega_n) := \Psi_k$. So we include the summation over modes k and Matsubara frequencies ω_n in a single object.

The action of the system is the sum of three terms

$$S = S_e + S_{ph} + S_{int} \qquad S_e = \sum_k \bar{\Psi}_k(-i\omega_n + \epsilon_k)\Psi_k \qquad S_{ph} = \sum_l \bar{\Phi}_p(-i\omega_n + \Omega_p)\Phi_p$$

$$S_{int} = \sum_{q,k} g_q \bar{\Psi}_{k+q}\Psi_k(\bar{\Phi}_{-q} + \Phi_q) = \sum_q g_q n_q(\bar{\Phi}_{-q} + \Phi_q) \quad \text{with} \quad n_q = \sum_k \bar{\Psi}_{k+q}\Psi_k$$

Here ϵ_k is the dispersion of the electrons, Ω_k the phonon dispersion and $g_q > 0$ the coupling constant for a momentum transfer q. The partition function then reads

$$Z = \int D(\bar{\Psi}, \Psi)D(\bar{\Phi}, \Phi)e^{-S}$$

a) The path integral over the phonons is a Gaussian integral with a linear shift. Calculate it and obtain a new action for the electrons.

$$Z = \int D(\bar{\Psi}, \Psi)e^{-S_{eff}} \quad \text{with} \quad S_{eff} = S_e - \sum_q \frac{g_q n_q g_{-q} n_{-q}}{-i\omega_n + \Omega_q} + \log(-i\omega_n + \Omega_q) \tag{3}$$

b) Discuss the temporal properties and the sign of this interaction. It is useful to analytically continue the new coupling to real frequencies $i\omega_n \to \omega + i\eta$.