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Quantum Field Theory for Condensed Matter Physics
Exercise 3

(Submission date: 22.11.19)

3.1 Thermodynamics of the harmonic oscillator (10+5+5=20 points)

We want to investigate the coherent state path integral representation of the thermodynamics of the harmonic oscillator. The partition function is

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \text{with} \quad H = \omega a^\dagger a\,. \tag{1}$$

In the lecture it will be shown that the coherent states path integral representation of a normal ordered Hamiltonian leads to the representation of the partition function

$$Z = \int D(\bar{\Psi}, \Psi) \exp\left\{ -\int_0^\beta d\tau \left[\bar{\Psi}(\tau)\partial_\tau \Psi(\tau) + H[\bar{\Psi}(\tau), \Psi(\tau)] \right] \right\}\,. \tag{2}$$

The integral has the boundary conditions $\Psi(\tau = 0) = \Psi(\tau = \beta)$ for bosons and $\Psi(\tau = 0) = -\Psi(\tau = \beta)$ for fermions. The same holds for the conjugate field.

a) Evaluate the path integral. To do this it is useful to expand the fields in a Fourier series on the interval $(0, \beta)$ in frequencies ω_n.
Hint: The Jacobian in the path integral turns out to be $\frac{1}{\beta^2}\Pi_{n=1}\frac{\omega_n^2}{\beta^2}$.
Note that $\sinh(x)/x = \Pi_n(1 + x^2/(\pi n)^2)$

b) Calculate from this the free energy F, the inner energy U.

c) Calculate the variance of energy and particle number by appropriate differentiation.
To which expectation values of $(\bar{\Psi}, \Psi)$ do they correspond?

3.2 Phase operator and phase-number uncertainty (3+6+6=15 points)

In analogy with complex numbers, we introduce the polar decomposition of the destruction

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}}, \tag{3}$$

with

$$\sqrt{\hat{n}} = \sum_{n=0}^{\infty} \sqrt{\hat{n}}|n\rangle\langle n|, \quad e^{i\hat{\phi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \tag{4}$$

a) Prove that $e^{i\hat{\phi}}$ is not unitary. What follows for $\hat{\phi}$?

b) Show that

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \tag{5}$$

is an eigenstate of the operator $e^{i\hat{\phi}}$ with eigenvalue $e^{i\phi}$. Draw an analogy with momentum eigenstates and position eigenstates. What can you guess for the commutator $[\hat{n}, \hat{\phi}]$?

c) Show that $[\hat{n}, e^{i\hat{\phi}}] = -e^{i\hat{\phi}}$. By Taylor expansion, demonstrate that this commutator is compatible with the commutation relation

$$[\hat{n}, \hat{\phi}] = i. \tag{6}$$

Convince yourself that the Heisenberg uncertainty relation for number and phase hence becomes

$$\Delta n \Delta\phi \geq 1/2. \tag{7}$$

****BONUS** 3.3 Path integral of the Harmonic oscillator** (8+10+4+4=26 points)

Many of the manipulation needed to perform calculation with the path integral can be learned with the harmonic oscillator. We will start out here by calculating the real space representation of the thermal density matrix.

$$\rho_{f,i} = \langle x_f | \, e^{-\beta H} \, |x_i\rangle \quad \text{with} \quad H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \tag{8}$$

a) Show that $\rho_{f,i}$ can be written as

$$\int_{x_i(0)}^{x_f(\beta)} \mathcal{D}x \, \exp\{-S_E\} \quad \text{with} \quad S_E = \int_0^{\beta} d\tau \frac{m}{2}(\dot{x}^2 + \omega^2 x)$$

b) Evaluate now the path integral. Shift for this $x \to x_c + \delta x$, where x_c satisfies $\frac{\delta S_E}{\delta x}\Big|_{x=x_c} = 0$, which is equivalent to the Euler-Lagrange equations. It is useful to take care of the boundary conditions in the path integral by choosing appropriate boundary conditions for x_c. Expand $\delta x(t)$ in a Fourier series in the interval $(0, \beta)$ and evaluate the path integral over δx.
One should end up with

$$\rho_{f,i} = \sqrt{\frac{m\omega}{2\pi \sinh(\beta\omega)}} \exp\left[-\frac{m\omega}{2}\left((x_f^2 + x_i^2)\coth(\beta\omega) - \frac{2x_i x_f}{\sinh(\beta\omega)}\right)\right] \tag{9}$$

Hint: $\sinh(x)/x = \Pi_n(1 + x^2/(\pi n)^2)$

c) Calculate the low temperature ($\beta\omega \gg 1$) and the high temperature ($\beta\omega \ll 1$) limit of the density matrix.

d) Calculate the partition function from eq. (9) by setting $x_i = x_f$ and integrating over all space.