Universität Bonn
Physikalisches Institut
Prof. Dr. J. Kroha, F. Meirinhos, T. Lappe, M. Kajan

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Quantum Field Theory for Condensed Matter Physics
Exercise 2

(Submission date: 04.11.19)

2.1 Grassmann Calculus (2+2+2+3+4+3+4+3+6=29 points)

In this problem we retrace the developments of the lecture and construct a coherent-state basis for fermions. Let us start by considering a single-particle state. In analogy to the bosonic case, we define the coherent state

$$\hat{c}\,|\Psi\rangle = \Psi\,|\Psi\rangle$$

The eigenvalues Ψ are *Grassmann numbers*, which anti-commute with other Grassmann numbers ($\Psi_i\Psi_j = -\Psi_j\Psi_i$), and with fermionic creation/annihilation operators.

a) Show that $\Psi^2 = 0$ and that the coherent states can be written in occupation-number basis as $|\Psi\rangle = |0\rangle - \Psi\,|1\rangle$.

b) Let us define also the left eigenstate of the creation operator as $\langle\bar{\Psi}|\,\hat{c}^\dagger = \langle\bar{\Psi}|\,\bar{\Psi}$. Calculate the overlap $\langle\bar{\Psi}|\Psi\rangle$.

The anti-commuting nature of Grassmann numbers leads to a particularly easy expansion of any function $f(\Psi)$, namely

$$f(\Psi) = f_0 + f_1\Psi\,. \tag{1}$$

Next, we introduce integration and differentiation on Grassmann numbers. Note that we define the derivative as acting from the left, which is a matter of convention.

$$\int \mathrm{d}\Psi\,\Psi = 1 = -\int \Psi\,\mathrm{d}\Psi, \qquad \int \mathrm{d}\Psi\;1 = 0, \qquad \partial_\Psi\Psi = 1, \qquad \partial_\Psi\bar{\Psi}\Psi = -\bar{\Psi}. \tag{2}$$

c) Generalize equation (1) to the case $f(\bar{\Psi},\Psi)$ and give the expansion of $e^{\bar{\Psi}\Psi}$.

d) Use equation (1) and (2) to show that

$$\int \mathrm{d}\bar{\Psi}\mathrm{d}\Psi e^{-a\bar{\Psi}\Psi} = a\,.$$

e) Show that the identity can be written as

$$\mathbb{1} = \int \mathrm{d}\bar{\Psi}\mathrm{d}\Psi\,|\Psi\rangle\langle\bar{\Psi}|\,e^{-\bar{\Psi}\Psi}\,.$$

We turn to the case of n single-particle states and define a set of Grassmann numbers $\{\Psi_i\}$. Eq. (1) generalizes analogously to c). Let A be a complex $n \times n$ matrix. Note that we use summation convention. We aim to compute the integral

$$\int \mathrm{d}\bar{\Psi}_1 \mathrm{d}\Psi_1 \mathrm{d}\bar{\Psi}_2 \mathrm{d}\Psi_2 ... \mathrm{d}\bar{\Psi}_n \mathrm{d}\Psi_n \, e^{-\bar{\Psi}_i A_{i,j} \Psi_j}.$$

f) Use the property discussed previously to simplify the exponential into a polynomial.

g) Order integration measure and polynomial appropriately, keeping track of the minus signs. *Hint*: The residual indices from the matrix inside the exponential can be worked out with

$$\Psi_{j_1}\Psi_{j_2}...\Psi_{j_n}\bar{\Psi}_{i_1}\bar{\Psi}_{i_2}...\bar{\Psi}_{i_n} = \epsilon_{j_1,j_2...j_n}\epsilon_{i_1,i_2,...i_n}\Psi_1\Psi_2...\Psi_n\bar{\Psi}_1\bar{\Psi}_2...\bar{\Psi}_n \, .$$

h) Show that the integral evaluates to

$$\frac{1}{n!}\epsilon_{j_1,j_2...j_n}\epsilon_{i_1,i_2,...i_n}A_{i_1,j_1}A_{i_2,j_2}...A_{i_n,j_n} = \det A.$$

i) Let $\{\bar{J}_i, J_i : i = 1...n\}$ be elements of the Grassmann algebra and M an invertible matrix. Calculate the integral

$$\int \mathrm{d}\bar{\Psi}_1 \mathrm{d}\Psi_1 \mathrm{d}\bar{\Psi}_2 \mathrm{d}\Psi_2 ... \mathrm{d}\bar{\Psi}_n \mathrm{d}\Psi_n \, e^{-\bar{\Psi}_i M_{i,j} \Psi_j} e^{\bar{J}_i \Psi_i + \bar{\Psi}_i J_i} \, .$$

2.2 Coherent States (2+2+2+2+2=10 points)

Coherent states have various interesting properties, which are resembled classical results. We will derive here a few examples for the harmonic oscillator. The position and momentum operators are given as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)\, .$$

a) Use Heisenbergs equation of motions to calculate the time dependence of the annihilation operator. Conclude the time dependence of the coherent states from this.

b) Calculate the time dependent average of the position operator $\hat{x}$ and the momentum $\hat{p}$ w.r.t. to the coherent states.

c) Calculate the time dependent variance Δx and Δp. What follows for the uncertainty relation of $\Delta x \Delta p$?

d) Calculate the variance of the particle number $\hat{n} = a^\dagger a$.

e) Calculate the action of the operator $e^{i\varphi\hat{n}}$ acting on a coherent state.