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Quantum Field Theory for Condensed Matter Physics
Exercise 1

(Submission date: 18.10.19)

1.1 Gaussian Integrals (2+2+4+3+4=15 points)

Consider the vector $\boldsymbol{x} \in \mathbb{R}^d$ and the symmetric, positive-definite matrix $A \in \mathbb{R}^{d\times d}$. Note that this implies that A is diagonalizable.

a) Calculate the following integral over $\mathbb{R}^d$: boundaries $x_i \in]-\infty, \infty[$:

$$\int d^d x \exp\left(-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}\right) .$$

b) Let $\boldsymbol{J}$ be a d-dimensional vector. Calculate the integral with the same boundaries:

$$I[J] = \int d^d x \exp\left(-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}} A\, \boldsymbol{x} + \boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{x}\right) .$$

c) Let f be an analytic function of $\boldsymbol{x}$, which grows slowly enough so that the following integral exists (e.g. polynomials). Calculate

$$\langle f(\boldsymbol{x})\rangle = \frac{1}{I[0]} \int d^d x f(\boldsymbol{x}) \exp\left(-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}} A\, \boldsymbol{x}\right) .$$

We can generalize these results to complex vectors $\boldsymbol{\phi} \in \mathbb{C}^d$ and a Hermitian, positive-definite matrices H by noting that the eigenvalues of H are real and positive, and that the d-dimensional, complex vectors $\boldsymbol{\phi}$ have $2\,d$ independent components. They can be represented either as real and imaginary part or by $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^\dagger$.

d) Calculate the following integral over $\mathbb{C}^d$:

$$Z[\boldsymbol{J}, \boldsymbol{J}^\dagger] = \int \mathrm{d}^d\phi \mathrm{d}^d\phi^* \exp\left(-\boldsymbol{\phi}^\dagger H \boldsymbol{\phi} + \boldsymbol{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \boldsymbol{J}\right) .$$

e) Let us define the expectation value of a function $f(\boldsymbol{\phi}, \boldsymbol{\phi}^\dagger)$ as

$$\langle f(\boldsymbol{\phi}, \boldsymbol{\phi}^\dagger)\rangle = \frac{1}{Z[0,0]} \int \mathrm{d}^d\phi \mathrm{d}^d\phi^*\ f(\boldsymbol{\phi}, \boldsymbol{\phi}^\dagger) \exp\left(-\boldsymbol{\phi}^\dagger H \boldsymbol{\phi}\right) .$$

Calculate the two expectation values

$$\langle \phi_i^* \phi_j \rangle \quad \text{and} \quad \langle \phi_i^* \phi_j^* \phi_k \phi_l \rangle .$$

1.2 Operators in second quantization (8+7=15 points)

We consider a system of N identical particles. The Hamiltonian is comprised of the kinetic energy $T = \sum_i \boldsymbol{p}_i^2/2m$, the single-particle potential $U(\boldsymbol{r})$, and a two-body interaction $V(\boldsymbol{r}_i, \boldsymbol{r}_j)$.

$$\mathcal{H} = \sum_i \frac{\boldsymbol{p_i}^2}{2m} + U(\boldsymbol{r}_i) + \frac{1}{2}\sum_{i \neq j} V(\boldsymbol{r_i}, \boldsymbol{r_j}) \tag{1}$$

a) Show that the second quantized form of (1), for fermions as well as for bosons, is given by

$$H = \sum_{\alpha,\beta} \left(T_{\alpha,\beta} + U_{\alpha,\beta}\right) a_\alpha^\dagger a_\beta + \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} V_{\alpha,\beta,\mu,\nu} a_\alpha^\dagger a_\beta^\dagger a_\nu a_\mu$$

where $\{|\alpha\rangle\}$ is a complete and orthonormal single-particle basis, and the matrix elements are given by

$$T_{\alpha,\beta} = \langle\alpha|\, T\, |\beta\rangle \quad , \quad U_{\alpha,\beta} = \langle\alpha|\, U\, |\beta\rangle \quad , \quad V_{\alpha,\beta,\mu,\nu} = \langle\alpha,\beta|\, V\, |\mu,\nu\rangle \ .$$

b) Now consider as the single-particle basis the momentum eigenstates of free particles in a volume v,

$$\phi_k(\boldsymbol{r}) = \frac{1}{\sqrt{v}}\, e^{i\boldsymbol{k}\boldsymbol{r}}$$

which fulfill the orthogonality relation

$$\int d^3r \phi_{\boldsymbol{k}}^*(\boldsymbol{r})\phi_{\boldsymbol{k}'}(\boldsymbol{r}) = \delta_{\boldsymbol{k},\boldsymbol{k}'}.$$

Assume translational invariance, $V(\boldsymbol{r}_i, \boldsymbol{r}_j) = V(\boldsymbol{r}_i - \boldsymbol{r}_j)$, to show that the momentum representation of (1) reads

$$H = \sum_{\boldsymbol{k}} \frac{(\hbar \boldsymbol{k})^2}{2m} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{1}{v}\sum_{\boldsymbol{k},\boldsymbol{k}'} U_{\boldsymbol{k}-\boldsymbol{k}'} a_{\boldsymbol{k}'}^\dagger a_{\boldsymbol{k}} + \frac{1}{2v}\sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{q}} a_{\boldsymbol{k}'+\boldsymbol{q}}^\dagger a_{\boldsymbol{k}-\boldsymbol{q}}^\dagger a_{\boldsymbol{k}} a_{\boldsymbol{k}'}$$

where $V_{\boldsymbol{q}}$ and $U_{\boldsymbol{q}}$ are the Fourier transforms of the single-particle potential and the two-body interaction, respectively.

1.3 Bosonic coherent states (3+2+5+5=15 points)

The coherent states $|\phi\rangle$ are defined as the eigenstates of the bosonic annihilation operators a_i for a particle in state i, $i = 1, 2, \ldots,$

$$a_i |\phi\rangle = \phi_i |\phi\rangle \quad , \quad \phi_i \in \mathbb{C}$$

Here we construct these eigenstates and, thereby, show their existence.

a) Since $|\phi\rangle$ is a state of the Fock space, it can be expanded as

$$|\phi\rangle = \sum_{n_1,n_2,\ldots} C_{n_1,n_2,\ldots} |n_1, n_2, \ldots\rangle \tag{2}$$

where $|n_1, n_2, \ldots\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \ldots |0\rangle$ are occupation number eigenstates. Show that $|\phi\rangle$ can be written as

$$|\phi\rangle \equiv |\phi\rangle_{\phi_1,\phi_2,\ldots} = \exp\left[\sum_i \phi_i a_i^\dagger\right] |0\rangle \ .$$

Hint: By applying the annihilation operator on (2) one can obtain a recursive relation for the coefficients $C_{n_1,n_2\ldots}$. Assume $C_{0,0\ldots} = 1$.

Note that this construction is valid for any choice of the set of eigenvalues $\{\phi_1, \phi_2, \ldots\}$, $\phi_i \in \mathbb{C}$, $i = 1, 2, \ldots,$ and that $|\phi\rangle$ is an eigenvector of all the a_i's.

b) Show that the action of the creation operator on $|\phi\rangle$ is given by

$$a_i^\dagger |\phi\rangle = \partial_{\phi_i} |\phi\rangle \quad \text{and} \quad \langle\phi| a_i = \partial_{\phi_i^*} \langle\phi| \ .$$

c) Show that the coherent states form a complete set of states in Fock space, i.e.,

$$\int \left[\prod_i \frac{\mathrm{d}\phi_i^* \mathrm{d}\phi_i}{\pi}\right] e^{-\sum_i \phi_i^* \phi_i} |\phi\rangle\langle\phi| = \mathbb{1} \, . \tag{3}$$

Hint: Following Schur's lemma, the LHS of (3) is proportional to $\mathbb{1}$ if it commutes with all operators in Fock space. Since all operators acting on Fock space can be expressed in terms of creation and annihilation operators, it is sufficient to show that the LHS commutes with them. The proportionality constant can be obtained by calculating the vacuum expectation value of the LHS.

d) Show that, by inserting the completeness relation (3), the trace of an operator $\hat{A}$ is converted to an integral over the coherent-state eigenvalues,

$$\mathrm{Tr} A = \int \left[\prod_i \frac{d\phi_i^* d\phi_i}{\pi}\right] e^{-\sum_i \phi_i^* \phi_i} \langle\phi| \hat{A} |\phi\rangle \ .$$