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# Theoretical Condensed Matter Physics
# Exercise 9

(Submission: 18.01.19, discussion: 21./23.01.19)

## 9.1 Proximity effect of a superconductor (10 points)

We consider an interface between a superconductor (S) and a normal metal (N) in the $y$-$z$ plane, see Fig 1. Within Ginzburg-Landau theory, the $S - N$ interface can be described by a free-energy functional where in the superconductor ($x > 0$) the temperature $T$ is below the critical temperature, that is, the reduced temperature is $\tau = (T - T_c)/T_c < 0$, while in the normal metal the parameter $\tau$ is $\tau = \tau_N > 0$ (temperature-independent). The Euler-Lagrange equation for the superconducting wave function (order parameter) was derived in the lecture from the position-dependent Ginzburg-Landau free energy as

$$-\xi_0^2 \, \nabla^2 \Psi + \tau \, \Psi + \frac{B}{A} \, |\Psi|^2 \, \Psi = 0 \ , \tag{1}$$

with the Ginzburg-Landau parameters $A$, $B$, a characteristic length scale $\xi_0$ and the reduced temperature $\tau = (T - T_c)/T_c$.

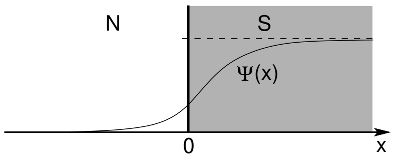


Fig. 1: Proximity effect.

a) Explain why the parameter $\tau$ should be chosen in the superconductor and in the normal metalas described above .

b) What are the boundary conditions on the order parameter $\Psi(x)$ deep in the superconductor ($x \to +\infty$), deep in the normal metal ($x \to -\infty$) and at the interface ($x = 0$)?

c) In the normal metal and near the interface the order parameter will be small. In these regions, we can, therefore, expand Eq. (9.1) up to linear order in $\Psi(x)$. Write down the linearized equation of motion in the superconductor near $x = 0$ and in the normal metal.

d) Determine $\Psi(x)$ in the normal metal and near the interface, with its value $\Psi(0)$ at the interface still an undetermined parameter. Determine the decay length $\lambda_N$ of the order parameter into the normal metal.
(Hint: For $x > 0$, the two elementary solutions can be chosen real, since there is no current flowing.) Linearize the solution in the region $x > 0$, since it is valid only for small $\Psi(x)$ (see c)).

e) Solving the nonlinear Euler-Lagrange equation for larger values $x > 0$ is difficult. Instead we assume for simplicity that $\Psi(x)$ continues to be a linear function for $x > 0$ and reaches the bulk value (as determined in a)) after the correlation length $\xi = \xi_0/\sqrt{|\tau|}$. With this assumption, determine the interface value $\Psi(0)$ as a function of temperature $\tau$.

We see that continuity enforces the wave function $\Psi(x)$ to extend a finite width into a normal metal. This is called the *proximity effect*. It means that Cooper pairs can propagate a finite "proximitiy length" $\lambda_N$ into the normal metal if the latter is close to a superconductor.

## 9.2 Magnetic susceptibility in Ginzburg-Landau theory (10 points)

We investigate the magnetic susceptibility of a ferromagnet, i.e., the response to a general, position-dependent magnetic field, near a ferromagnetic phase transition. The Ginzburg-Landau free energy for the magnetization $m(\mathbf{r})$ of a ferromagnet in a magnetic field $H(\mathbf{r})$ reads,

$$\mathcal{F}\{m(\mathbf{r}),\, H(\mathbf{r})\} = \frac{1}{a}\int \mathrm{d}^3\mathbf{r}\, \left[\xi_0^2\,(\nabla m(\mathbf{r}))^2 + \tau\, m(\mathbf{r})^2 + \frac{b}{2a}\, m(\mathbf{r})^4 - H(\mathbf{r})m(\mathbf{r})\right], \tag{2}$$

with the Ginzburg-Landau parameters $a$, $b$, the reduced temperature $\tau = (T - T_c)/T_c$, and a stiffness length $\xi_0$. The last term in this expression represents the energy of the system due to the magnetic field. For simplicity we consider only the magnetization component parallel to the magnetic field.

a) Without magnetic field ($H(\mathbf{r}) = 0$), and without spatial boundaries of the system, the magnetization will be homogeneous, $m(\mathbf{r}) = m_0 = \mathit{const}$. Determine first the zero-field magnetization $m_0$ as a function of temperature $\tau$.

b) Derive the Euler-Lagrange equation for the magnetization $m(\mathbf{r})$ in a position-dependent magnetic field $H(\mathbf{r})$.

c) Because of the gradient term in the free energy ("stiffness"), a magnetic field acting at position $\mathbf{r}'$ will also lead to a change of the magnetization at a different position $\mathbf{r}$. The susceptibility is defined as the change of $m(\mathbf{r})$ due to a *small* magnetic field $H(\mathbf{r}')$ and is, therefore, a *non-local* quantity:

$$\chi(\mathbf{r} - \mathbf{r}') = \frac{\delta m(\mathbf{r})}{\delta H(\mathbf{r}')}\,. \tag{3}$$

Derive the equation for $\chi(\mathbf{r} - \mathbf{r}')$ by taking the $H$-field derivative of the Euler-Lagrange equation. Now take the limit of vanishing magnetic field using the result of problem a) and show that in this limit the equation for $\chi(\mathbf{r} - \mathbf{r}')$ takes the form

$$\left(-\nabla^2 + \frac{1}{\xi_{\lessgtr}^2}\right)\chi(\mathbf{r} - \mathbf{r}') = \frac{1}{\xi_0^2}\,\delta(\mathbf{r} - \mathbf{r}'), \tag{4}$$

where $\xi_>$ or $\xi_<$ is the *correlation length* in the ferromagnetic ($\tau < 0$) or in the paramagnetic ($\tau > 0$) phase, respectively. Give explicitly the expressions for $\xi_<$ and $\xi_>$.

d) Solve this equation by Fourier transform for the susceptibility $\chi(\mathbf{q})$ at wave vector $\mathbf{q}$.

e) Fourier transform $\chi(\mathbf{q})$ back to position space (complex contour integration). How does $\chi(\mathbf{r} - \mathbf{r}')$ behave as a function of the distance coordinate $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}'$? Why is $\xi_{\lessgtr}$ then called correlation length?

f) How does the homogeneous part of the susceptibility, $\chi(\mathbf{q} = 0, \tau)$ behave at the phase transition, $\tau \to 0^{\pm 0}$? What are, in Ginzburg-Landau theory, the critical exponents of the correlation length $\xi(\tau)$ and of the susceptibility $\chi(\mathbf{q} = 0, \tau)$?