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14.12.2018
Winter semester 2018/19
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Theoretical Condensed Matter Physics
Exercise 8

(Submission: 11.01.19, discussion: 14./16.01.19)

8.1 Debye model (acoustic phonons): continuation (5 points)

Consider the expression for the root-mean-square fluctuations of the i-th basis atom in the unit cell

$$\langle x_i^2 \rangle \approx \int_0^{+\infty} d\varepsilon\, g(\varepsilon) \frac{\hbar^2}{m_i^* \, \varepsilon} \left(n_B(\varepsilon) + \frac{1}{2} \right).$$

Evaluate its *thermal* and *quantum* contributions separately, within the Debye model. Considering the phenomenological Lindemann melting criterion, by which a solid *melts* when the atom position fluctuations exceed a fraction of the lattice spacing, what could you say about the existence of solids in $d = 1, 2, 3$ dimensions at *zero* and *finite* T?

Hint: $\int_0^x du \, \frac{u^{d-2}}{e^u - 1} = \begin{cases} \frac{x^{d-2}}{d-2} & x \to 0, \\ \zeta(d-1) & x \to \infty. \end{cases}$

8.2 Phonons in a metal: Kohn Anomaly (10 points)

In metals one expects that the harmonic interactions between the ions of a lattice are not only between nearest neighbours, but are long-ranged. The physical origin is a displacement of an ionic charge that induces an electric charge polarization in the electron sea which is long-ranged and oscillatory in space. This oscillatory polarization acts on the other ions in the lattice. The oscillations are induced by the fact that the electronic states are filled up to the Fermi wave number k_F and the wave number of the resulting oscillatory ion-ion potential is $2k_F$ (Friedel oscillations).

We now consider a one-dimensional linear chain of ions in an electron sea with the lattice constant a. The force constant between an ion on lattice site i and an ion on lattice site j is

$$\kappa_{ij} = \kappa_0 \frac{\sin 2k_F a(i-j)}{k_F a(i-j)}$$

a) Write down the Lagrange function for the system of ions and derive the equations of motion.

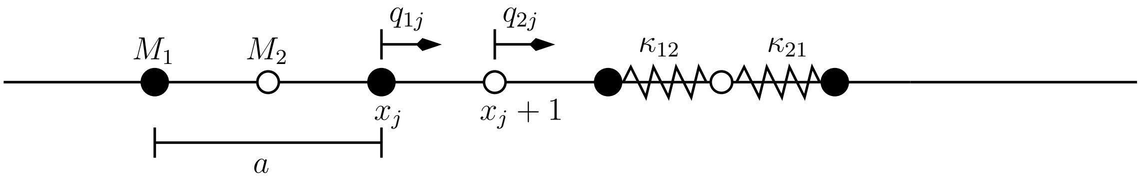
b) Make an ansatz of plane wave solutions for the ion displacements:

$$q_j(t) = q_0 \, e^{i(k\,x - \omega\, t)}, \qquad x_j = j \cdot a \tag{1}$$

Derive an expression for $\omega(k)^2$ and $\frac{\partial \omega^2}{\partial k}$ and show that $\omega(k)^2$ has a cusp (divergent slope) at $k = 2k_F$. This effect is called Kohn anomaly after Walter Kohn, who predicted it in 1959.

8.3 Optical Phonons (10 points)

Consider a one-dimensional harmonic chain with two different atoms in the primitive cell,



where $x_j = a \cdot j$ is the position of the jth primitive cell and q_{sj} is the elongation of the sth atom in the jth cell. The Hamiltonian of the system is given by

$$H = \sum_{js} \frac{1}{2} M_s \dot{q}_s(x_j) + \frac{1}{2} \sum_j \left[\kappa_{12}(q_{2j} - q_{1j})^2 + \kappa_{21}(q_{1j} - q_{2j-1})^2\right]$$

For simplicity we assume $\kappa_{12} = \kappa_{21} = \kappa$.

a) Write down the equations of motion. Use the ansatz $q_{sj} = q_s^0 e^{i(k\,a\,j - \omega\,t)}$ with $s = 1, 2$ to derive the following equation for the frequencies of the eigenmodes (phonons):

$$\omega^2 = \frac{\kappa}{M^*} \pm \sqrt{\left(\frac{\kappa}{M^*}\right)^2 - 2\frac{\kappa^2}{M_1 M_2}\,(1 - \cos ka)},$$

and $M^* = \frac{M_1 M_2}{M_1 + M_2}$ is the reduced mass.

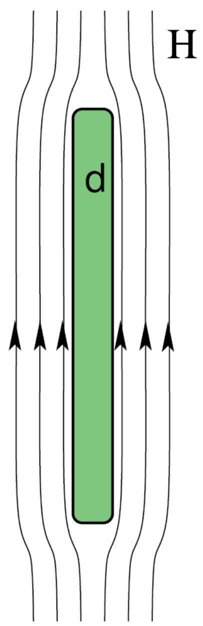
b) Calculate $\omega(k)^2$ for $k \to 0$ and $k \to \pm\pi$ and sketch its behaviour for the full Brillouin zone $[-\pi, \pi]$. Identify the acoustic and the optical branch.

c) Discuss the limit $M_1 = M_2$ and show that this limit results in the acoustic phonon dispersion with lattice constant $a/2$.

8.4 Meissner effect (10 points)

The London equations of superconductivity are

$$\frac{\partial}{\partial t}(\Lambda \boldsymbol{j}) = \boldsymbol{E} \qquad \nabla \times (\Lambda \boldsymbol{j}) = -\boldsymbol{H} \qquad \Lambda = \frac{m}{n_s e^2}$$



Consider a thin superconducting slab of thickness d, infinitely extended in y and z directions, in a uniform, static magnetic field $\boldsymbol{H} = H_0\,\hat{e}_z$, parallel to the slab surface.

a) Use the Maxwell equations to calculate the magnetic field $\boldsymbol{H}(\boldsymbol{x})$ inside the slab. Show that the field enters the slab only on a length scale λ_L (London penetration depth). Draw the result.

b) Calculate also the current density inside the slab and draw the result.

c) Calculate the average magnetization of the slab,

$$\langle H(x) \rangle = \frac{1}{d} \int_{-d/2}^{d/2} dx\, H(x)$$

and draw it as a function of d.