Universität Bonn
Physikalisches Institut
Prof. Dr. H. Kroha, T. Lappe, H. Qiu, F. Meirinhos

09.11.2018
Winter semester 2018/19
http://www.kroha.uni-bonn.de

Theoretical Condensed Matter Physics
Exercise 4

(Submission: 23.11.18, discussion: 26./28.11.18)

4.1 Thermodynamic properties of a Fermi liquid at low T (15 points)

In the lecture the concept of quasiparticles was introduced. The thermodynamic properties of a Fermi liquid are determined by the properties of the quasiparticle gas. For one quasiparticle above the ground state the energy can be linearly approximated by $\varepsilon_{\mathbf{k}} = v_F\,(k - k_F)$ with $v_F = k_F/m^*$. For more than one excited quasiparticle, interactions between the quasiparticles come into play. For the interaction-induced contribution to the quasiparticle energy $E_{\mathbf{k}\sigma} = \varepsilon_{\mathbf{k}} + \delta E_{\mathbf{k}\sigma}$ we derived,

$$\delta E_{\mathbf{k}\sigma} \approx \frac{1}{N_F}\left[F_0^s\,\delta n + \sigma\,F_0^a\,\delta n_s + \frac{1}{k_F^2}\,F_1^s\,\mathbf{k}\,\delta\mathbf{g}\right] , \tag{1}$$

with $F_l^{s,a}$ the dimensionless Landau parameters. $\delta n = \sum_{\mathbf{k}\sigma}(n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma})$ is the density of excited quasiparticles and $\delta n_s = \sum_{\mathbf{k}\sigma}\sigma(n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma})$ the spin density of excited quasiparticles. For the particle current density the relation $\mathbf{j} = (1/m^*)\mathbf{g} = (1/m^*)\sum_{\mathbf{k}}\mathbf{k}\,n_{\mathbf{k}\sigma}$ holds, where $\mathbf{g}$ is the momentum density.

The internal energy of the system is $U = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}\sigma}\,n(E_{\mathbf{k}\sigma})$, with $n(E)$ the Fermi-Dirac distribution. By differentiation with respect to the temperature T we can calculate the specific heat,

$$c_V = \frac{\partial}{\partial T}\sum_{\mathbf{k}\sigma} E_{\mathbf{k}\sigma}\,n(E_{\mathbf{k}\sigma}) . \tag{2}$$

(a) Show that

$$c_V = \frac{1}{T}\sum_{\mathbf{k}\sigma} E^2_{\mathbf{k}\sigma}\left(-\frac{\partial n}{\partial E_{\mathbf{k}\sigma}}\right) + \sum_{\mathbf{k}\sigma}\frac{\partial E_{\mathbf{k}\sigma}}{\partial T}n \tag{3}$$

(b) Show that in leading order of T the specific heart is given by

$$c_V = \frac{\pi^2}{3}N_F T$$

with $N_F = (k_F m^*)/\pi^2$ the density of states at the Fermi level.
Hint: Start with equation (2), transform the k-sum into an integral over ε and use a Sommerfeld expansion for the integral of the type $\int d\varepsilon F(\varepsilon)n(\varepsilon)$.

Therefore, for low T the specific heat is proportional to the temperature for all metals. The second term in (3) would lead to a correction $\delta c_V \propto T^2\ln(T/E_F)$.

In the lecture the spin susceptibility χ was derived by considering the system under the influence of a magnetic field. It was shown that χ is related to the Landau parameter F_0^a as

$$\chi = \mu_M^2\frac{N_F}{1+F_0^a}$$

In the following we want to derive analogously the relation between the compressibility κ and F_0^s. Therefore, ee consider a system of Volume V under pressure P.

(c) The compressibility is defined as $\kappa = -\frac{1}{V}\frac{dV}{dP}$. Show that κ can be expressed by

$$\kappa = \frac{1}{n^2}\frac{dn}{d\mu}$$

Hint: Use $dG = -SdT - Nd\mu + VdP = 0$ and the fact that the chemical potential μ is intensive, i.e, only depends on the particle density $n = N/V$, that is $\mu(N/V)$.

Due to the change of the particle density the energy is shifted by μ and the particle density is therefore changed by $\delta n = \frac{1}{V}\sum_{\mathbf{k}\sigma}\left(n\left(E_{\mathbf{k}\sigma\sigma}(\mu)\right) - n\left(E_{\mathbf{k}\sigma}(0)\right)\right)$ Since the energy itself depends on the particle density, δn results in an additional correction $\delta E_{\mathbf{k}\sigma}$. Thus, the energy reads

$$E_{\mathbf{k}\sigma}(\mu) = E_{\mathbf{k}\sigma}(0) - \mu + \delta E_{\mathbf{k}\sigma}.$$

(d) Expand $n(E_{\mathbf{k}\sigma}(\mu))$ for small μ around $\mu = 0$ up to second order.

(e) $\delta E_{\mathbf{k}\sigma}$ is given by the first term in (1). Plug this into your expansion and finally derive

$$\kappa = \frac{1}{n^2}\frac{N_F}{1 + F_0^s}$$

4.2 Quasiparticle distribution in equilibrium (5 points)

The Boltzmann equation reads (see lecture)

$$\frac{\partial n_{\mathbf{k}\sigma}}{\partial t} + \mathbf{v}_{\mathbf{k}} \cdot \frac{\partial n_{\mathbf{k}\sigma}}{\partial \mathbf{r}} - \frac{\partial E_{\mathbf{k}\sigma}}{\partial \mathbf{r}}\frac{\partial n^0_{\mathbf{k}\sigma}}{\partial \mathbf{k}} = I\left\{n_{\mathbf{k}\sigma}\right\}$$

with the collision integral

$$I\{n_{\mathbf{k}\sigma}\} = -\sum_{\substack{\mathbf{k}_1,\mathbf{k}',\mathbf{k}'_1\\ \sigma_1,\sigma',\sigma'_1}} f_{\substack{\mathbf{k}'\mathbf{k}'_1,\mathbf{k}\mathbf{k}_1\\ \sigma'\sigma'_1,\sigma\sigma_1}}\left[n_{\mathbf{k}\sigma}n_{\mathbf{k}_1\sigma_1}\left(1 - n_{\mathbf{k}'\sigma'}\right)\left(1 - n_{\mathbf{k}'_1\sigma'_1}\right) - \left(1 - n_{\mathbf{k}\sigma}\right)\left(1 - n_{\mathbf{k}_1\sigma_1}\right)n_{\mathbf{k}'\sigma'}n_{\mathbf{k}'_1\sigma'_1}\right]$$
$$\cdot\,\delta\left(E_{\mathbf{k}\sigma} + E_{\mathbf{k}_1\sigma_1} - \left(E_{\mathbf{k}'\sigma'} + E_{\mathbf{k}'_1\sigma'_1}\right)\right)$$

Show that for the distribution functions

$$n^0_{\mathbf{k}\sigma} = \frac{1}{e^{\frac{E_{\mathbf{k}\sigma}}{k_B T}} \pm 1}$$

('+' for fermions, '−' for bosons) the collision integral vanishes: $I\left\{n^0_{\mathbf{k}\sigma} = 0\right\}$. Conclude that $n^0_{\mathbf{k}\sigma}$ is an $\mathbf{r}$- and t-independent solution of the Boltzmann equation and that, therefore, $n^0_{\mathbf{k}\sigma}$ is the equilibrium distribution. Are other equilibrium distributions mathematically possible?