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Theoretical Condensed Matter Physics
Exercise 10

(Submission: 25.01.19, discussion: 28./30.01.19)

10.1 Bardeen–Cooper–Schrieffer theory (10 points)

The purpose of this exercise is to diagonalize the BCS Hamiltonian and derive the so-called BCS *gap equation*. As introduced in the lecture, the attractive phonon-mediated interaction between electrons allows the formation of bound states (Cooper pairs) in the superconductor below some critical temperature T_c. The existence of an energy gap when $T < T_c$ protects Cooper pairs from to being thermally excited and destroyed. We start with the so-called *pairing* or *reduced Hamiltonian*, which includes the terms that are relevant for superconductivity,

$$H_R = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow},$$

where for instance $c^\dagger_{\mathbf{k}\uparrow}$ and $c_{\mathbf{k}\uparrow}$ are the creation/destruction operators of a particle with quantum number $\mathbf{k}\uparrow$. The single particle energy $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$ is measured from the Fermi energy. The effective attractive electron-electron interaction is $V_{\mathbf{kl}}$.

a) Derive the so called model Hamiltonian H_M from the reduced Hamiltonian H_R. The result is,

$$H_M = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} \left(c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} b_{\mathbf{l}} + b^\dagger_{\mathbf{k}} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} - b^\dagger_{\mathbf{k}} b_{\mathbf{l}} \right)$$

where $b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle_{av}$. To do this, apply $c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = b_{\mathbf{k}} + (c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}})$ to the second term in H_R and neglect the terms are bilinear in $c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}}$ (fluctuations). This kind of decomposition is sometimes called mean-field approximation.

b) Bogoliubov and Valatin pointed out independently that to diagonalize the above model Hamiltonian, one can apply a linear transformation (now known as the Bogoliubov-Valatin transformation),

$$c_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1} \qquad c^\dagger_{-\mathbf{k}\downarrow} = -v^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + u_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1}$$

where the $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$.

Derive the diagonalized Hamiltonian $H_{M,\text{Bogoliubov}}$ from the H_M. The resulting Hamiltonian should look like, $H_{M,\text{Bogoliubov}} = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} b_{\mathbf{k}}) + \sum_{\mathbf{k}} E_{\mathbf{k}} \left(\gamma^\dagger_{\mathbf{k}0} \gamma_{\mathbf{k}0} + \gamma^\dagger_{\mathbf{k}1} \gamma_{\mathbf{k}1} \right)$, where $\Delta_{\mathbf{k}} = -\sum_{\mathbf{l}} V_{\mathbf{kl}} \langle c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \rangle_{av}$, $E_{\mathbf{k}} = (\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2)^{1/2}$. Explain the physical meaning of $E_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$.

c) From the Bogoliubov transformation we can derive the BCS gap equation. First rewrite $\Delta_{\mathbf{k}}$ in terms of $\gamma_{\mathbf{k}1}\ \gamma_{\mathbf{k}2}$ and then neglect the off-diagonal terms $\gamma^\dagger_{\mathbf{k}0} \gamma^\dagger_{\mathbf{k}1}$ and $\gamma_{\mathbf{k}1} \gamma_{\mathbf{k}0}$, as they do not contribute to the averages. Use $\left\langle \gamma^\dagger_{\mathbf{k}0} \gamma_{\mathbf{k}0} \right\rangle = \left\langle \gamma^\dagger_{\mathbf{k}1} \gamma_{\mathbf{k}1} \right\rangle = f(E_{\mathbf{k}})$ to obtain the gap equation. Explain why $f(E)$ is the Fermi-Dirac distribution.

10.2 Domain walls in Ginzburg-Landau theory (10 points)

Earlier we studied the behaviour of a position-dependent order parameter in connection to the boundary condition that it should vanish at an *interface*. In this exercise, we would like to see what happens if we instead explicitly impose inhomogeneous boundary conditions at $\pm\infty$.

Large deviations from equilibrium are costly for $T < T_c$. In magnetic systems, i.e. the Ising model, they can occur, however, at so-called *domain walls* or *solitons*, where the order parameter changes sign from $-m_0$ to $+m_0$. In one dimension, the Ginzburg-Landau free energy for the magnetization $m(r)$ of a ferromagnet in a magnetic field $H(r)$ reads

$$\mathcal{F}\{m(r),\, H(r)\} = \frac{1}{a}\int \mathrm{d}r \left[\xi_0^2 \left(\frac{\partial}{\partial r} m(r)\right)^2 + \tau\, m(r)^2 + \frac{b}{2a}\, m(r)^4 - H(r)m(r)\right], \tag{1}$$

again with the usual parameters a, b, the reduced temperature $\tau = (T - T_c)/T_c$, and a stiffness length ξ_0. The boundary conditions we would like to investigate are $m(r \to \pm\infty) = \pm m_0$, where m_0 is again the *uniform* solution, and $m'(r \to \pm\infty) = 0$.

a) Repeat the derivation of the Euler-Lagrange equation for the order parameter $m(r)$. How are the prefactors in this *real* equation different from the case of a *complex* order parameter $\psi(r)$ as for superconductors?

b) Write the Euler-Lagrange equation in analogy to Newton's law of motion as

$$m''(r) = -\frac{\mathrm{d}\mathcal{V}[m]}{\mathrm{d}m}, \tag{2}$$

where m plays now the role of a "trajectory" parameterized by "time" r. Find the corresponding "kinetic energy" $\mathcal{T}$ and the "potential" $\mathcal{V}[m]$, and use them to express the conservation of the *total* energy $\mathcal{E}$.

c) Integrate $\mathrm{d}\mathcal{E}[m(\tilde{r})]/\mathrm{d}\tilde{r}$ from $-\infty$ to r, while keeping in mind the boundary conditions. Use your result to find an equation for $m'(r)$.

d) Solve the resulting equation from c) by separation of variables. Make a sketch of your solution. What is the meaning of the integration constant r_0 from the spatial integral?