**Universität Bonn** 18 November, 2016
**Physikalisches Institut**
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http://www.kroha.uni-bonn.de/teaching

# Special Topics in Condensed Matter Theory
# Winter term 2016/17

## Exercise 5

(Solutions due on 23 December, 2016)

In the lecture we learnt the Feynman diagram technique for imaginary time Green's functions or Matsubara Green's functions, this corresponds to finite temperature quantum systems, e.g. electron gas. Here we apply it and consider an important second order diagram for particle-particle interaction.

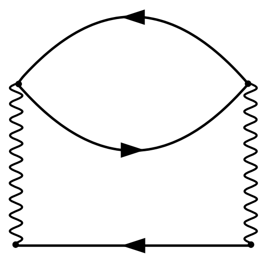
**6.1 Feynman Diagrams of Imaginary Time Green's Functions** (10 points)
The Hamiltonian of a many-electron system reads,

$$H = \sum_{\mathbf{p}\sigma}(\epsilon_p - \mu)c^\dagger_{\mathbf{p}\sigma}c_{\mathbf{p}\sigma} + \sum_{\sigma\sigma'}\int d^3r_1 \int d^3r_2 c^\dagger_{\mathbf{r}_1\sigma}c_{\mathbf{r}_2\sigma}W(|\mathbf{r}_1 - \mathbf{r}_2|)c^\dagger_{\mathbf{r}_{2\sigma'}}c_{\mathbf{r}_2\sigma'}, \quad (1)$$

where we consider a continuous system with single-particle energy $\epsilon_\mathbf{p}$ and chemical potential $\mu$, with a particle-particle interaction strength $W(|\mathbf{r}_1 - \mathbf{r}_2|)$.

a) Consider the one-particle (two-point) Matsubara Green's function for Fermions $G(\mathbf{k}, ik_n)$, where $\mathbf{k}$ denotes the momentum and $k_n$ denotes the Matsubara frequency. Write down Fourier space Feynman rules for particle-particle interaction, i.e. for the Hamiltonian in Eq. (1).

b) Draw all Feynman diagrams up to second order in interaction strength $W(\mathbf{q})$. Figure out which parts are belong to the self-energy $\Sigma(\mathbf{k}, ik_n)$. Label the diagram below with appropriate momentum and Matsubara Frequencies, it is usually called *pair-bubble self energy.*



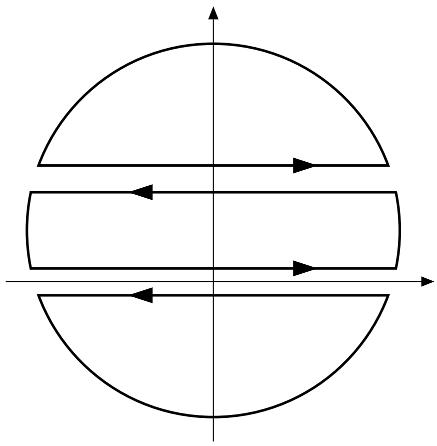
c) Use Fourier space Feynman rules to write down the explicitly formula of the *pair-bubble self energy.* One should reach something like,

$$\Sigma^P_\sigma(\mathbf{k}, ik_n) = \frac{1}{\beta}\sum_{iq_n}\int \frac{d^3q}{(2\pi)^3}W(\mathbf{q})^2\Pi^0(\mathbf{q}, iq_n)G^0_\sigma(\mathbf{k}-\mathbf{q}, ik_n - iq_n), \quad (2)$$

draw $\Pi^0(\mathbf{q}, iq_n)$ into Feynman diagram. This diagram corresponds to so called *pair-bubble* and also appears when one calculates the retarded density-density correlation function $\chi^R(\mathbf{q}, \omega)$ (this may remind you the problem sheet 1).

## 6.2 Lindhard function (10 points)

a) Perform the Matsubara summation in $\Pi^0(\mathbf{q}, iq_n)$, remind yourself that $iq_n$ is a Bosonic Matsubara frequency (why?). Before doing so, just finish the graph below, label the axis, place the poles and argue where are the branch cuts. [Hint: The free Green's function is $G^0_\sigma(\mathbf{p}, ip_n) = 1/(ip_n - (\epsilon_\mathbf{p} - \mu))$]



b) Now we analytically continue the $\Pi^0(\mathbf{q}, iq_n)$ to its retarded component $\Pi^R(\mathbf{q}, \omega)$ ($\sim \chi^R(\mathbf{q}, \omega)$). Write down the relation between $\Pi^0(\mathbf{q}, iq_n)$ and $\Pi^R(\mathbf{q}, \omega)$. Show that for $\omega = 0$ (the static case), we have,

$$\Pi^R(\mathbf{q}, 0) = 2 \int \frac{dk}{(2\pi)^3} \frac{f(\epsilon_{\mathbf{k}+\mathbf{q}} - \mu) - f(\epsilon_\mathbf{k} - \mu)}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_\mathbf{k}}$$
$$\overset{T\to 0}{=} 2 \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(\mu - \epsilon_{\mathbf{k}+\mathbf{q}/2}) - \Theta(\mu - \epsilon_{\mathbf{k}-\mathbf{q}/2})}{\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2}}$$

c) For small momentum transfer $|\mathbf{q}|$, the analytic solution of $\Pi^R(\mathbf{q}, 0)$ is,

$$\Pi^R(\mathbf{q}, 0) = \begin{cases} \frac{m}{\pi\epsilon_F} \frac{1}{x} \ln\left|\frac{1-x}{1+x}\right| & , d = 1 \\ -q_{TF}^2 \frac{1}{2}\left(1 + \frac{1-x^2}{2x} \ln\left|\frac{1-x}{1+x}\right|\right) & , d = 3 \end{cases} \tag{3}$$

where $x = \mathbf{q}/2k_F$, $k_F$ is the Fermi momentum. $\epsilon_F$ is the Fermi energy. And in $d = 3$ dimensions, $q_{TF} = \sqrt{4e^2 m\epsilon_F/\pi}$ is the so-called Thomas-Fermi wave number. $\Pi^R(\mathbf{q}, 0)$ is called the *Lindhard function*. Sketch "$\Pi^R(\mathbf{q}, 0)$ vs. $x$" for $1d$ and $3d$ solutions.

# 1 Solution

6.1 b)Since the self-energy only includes the one particle irreducible diagrams without two out legs.

6.2 a) [1]

$$\begin{aligned}\Pi^0(q, iq_n) =&\frac{-2}{\beta}\sum_{p_n}\int\frac{dp}{(2\pi)^3}G^0_{\sigma'}(p, ip_n)G^0_{\sigma'}(p+q, ip_n+iq_n)\\ =&-2\int\frac{dp}{(2\pi)^3}(-\oint_C\frac{dz}{2\pi i}n_F(z)G^0(p,z)G^0(p+q, z+iq_n)\end{aligned}$$

The contour in the problem sheet is somewhat misleading (sorry for this). Here we first perform the Matsubara summation, we have two simple poles on the complex plane at $z_1 = \xi_p$, $z_2 = \xi_{p+q} - iq_n$, since the free Green's function is $G^0_\sigma(p, ip_n) = 1/(ip_n - \xi_p)$. Use rule for simple pole (Bruus 11.57),

$$S^F_0(z) = \sum_j \text{Res}_{z=z_j}[g_0(z)]n_F(z_j)e^{z\tau} \tag{4}$$

we have,

$$\begin{aligned}\Pi^0(q, iq_n) =&2\int\frac{dp}{(2\pi)^3}\frac{1}{2\pi i}(\text{Res}_{z=\xi_p}[n_F(z)G^0(p,z)G^0(p+q, z+iq_n)]\\ &+\text{Res}_{z=\xi_{p+q}-iq_n}[n_F(z)G^0(p,z)G^0(p+q, z+iq_n)])\end{aligned}$$

Then use the residue theorem,

$$\begin{aligned}\Pi^0(q, iq_n) =&2\int\frac{dp}{(2\pi)^3}\frac{1}{2\pi i}[n_F(z)2\pi i G^0(p+q, \xi_p+iq_n)]\\ &+[n_F(\xi_{p+q}-iq_n)G^0(p, \xi_{p+q}-iq_n)2\pi i]\end{aligned}$$

Now it's time to substitute the expression of the Green's functions, i.e. $G^0$,

$$\begin{aligned}=&2\int\frac{dp}{(2\pi)^3}\left(n_F(\xi_p)G^0(p+q, \xi_p+iq_n) + n_F(\xi_{p+q}-iq_n)G^0(p, \xi_{p+q}-iq_n)\right)\\ =&2\int\frac{dp}{(2\pi)^3}\left(n(\xi_p)\frac{1}{\xi_p+iq_n-\xi_{p+q}} + n_F(\xi_{p+q}-iq_n)\frac{1}{\xi_p-\xi_{p+q}+iq_n}\right)\\ =&2\int\frac{dp}{(2\pi)^3}\left(n(\xi_p)\frac{1}{\xi_p+iq_n-\xi_{p+q}} + \frac{1}{e^{\beta\xi_{p+q}}e^{-\beta iq_n}+1}\frac{1}{\xi_p-\xi_{p+q}+iq_n}\right)\end{aligned}$$

The $iq_n$ is a Bonsonic Matsubara frequency, $iq_n = \dfrac{2n\pi}{\beta}$ which means $e^{\beta iq_n} = e^{i2n\pi} = 1$

$$\Pi^0(q, iq_n) =2\int\frac{dp}{(2\pi)^3}\left(\frac{n_F(\xi_p)+n_F(\xi_{p+q})}{iq_n+\xi_p-\xi_{p+q}}\right)$$

If we represent the Fermi distribution by $f(\xi)$ instead of $n_F(\xi)$ and consider the retarded part, we finally reach the Lindhard function,

$$\Pi^{0R}(q,\omega) = \Pi^0(q, \omega+i0^+) = 2\int\frac{dp}{(2\pi)^3}\left(\frac{f(\xi_p)+f(\xi_{p+q})}{\omega+\xi_p-\xi_{p+q}+i0^+}\right) \tag{5}$$

# References

[1] H. Bruus and K. Flensberg. *Many-body quantum theory in condensed matter physics: an introduction*. Oxford University Press, 2004.