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Special Topics in Condensed Matter Theory Winter term 2016/17

Exercise 2

(Solutions due on 11 November, 2016)

1.1 Green's functions for noninteracting electrons (15 points)

In the lecture the position dependent Green's function was defined. In the same way one can define a momentum dependent retarded Green's function,

$$G^R_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\frac{1}{Z_G}\mathrm{tr}\left\{\mathrm{e}^{-\beta(H-\mu N)}\left[c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')\right]_+\right\} \tag{1}$$

The advanced and time-ordered momentum dependent Green's functions are defined in analogy to the lecture. Here we will consider a system of noninteracting electrons,

$$\mathcal{H}_0 = H_0 - \mu N = \sum_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}} - \mu)c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}$$

a) Determine the time-dependence of $c_{\mathbf{k}\sigma}(t)$ and $c^\dagger_{\mathbf{k}\sigma}(t')$ for the noninteracting system $\mathcal{H}_0$ by using the equation of motion for a Heisenberg operator.

b) Compute the retarded Green's function (1) for the noninteracting system using the results of b).

$$G^{(0)R}_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\mathrm{e}^{-\mathrm{i}(\epsilon_{\mathbf{k}}-\mu)(t-t')} = G^{(0)R}_{\mathbf{k}\sigma}(t-t') \tag{2}$$

c) Derive the Fourier transform

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d(t-t')\, G^{(0)R}_{\mathbf{k}\sigma}(t-t')\,\mathrm{e}^{\mathrm{i}\omega(t-t')} = \frac{1}{\omega-(\epsilon_{\mathbf{k}}-\mu)+\mathrm{i}\eta}\ , \tag{3}$$

where $\eta > 0$ is an infinitesimal, positive number.

Hint: Observe and discuss that an infinitesimal imaginary part $i\eta$ must be introduced to the energy in order to make the integral convergent.

In an analogous manner, calculate the expression for the noninteracting, advanced Green's function $G^{(0)A}_{\mathbf{k}\sigma}(\omega)$.

d) The Green's function can also be derived from its equation of motion. Using the Heisenberg equation of motion for the operators $c_{\mathbf{k}\sigma}(t)$ or $c^\dagger_{\mathbf{k}\sigma}(t')$, show that the retarded as well as the advanced, noninteracting Green's function obeys the same equation of motion,

$$(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{H}_0)G^{(0)R/A}_{\mathbf{k}\sigma}(t-t') = \delta(t-t')\ ,$$

i.e. a "Schrödinger equation with $\delta-$Inhomogeneity".

e) Fourier transform the equation of motion of d) to energy space and solve it to obtain the retarded and advanced Green's functions in momentum and energy space $(\mathbf{k}, \omega)$. Discuss how the *boundary condition* of causality (retarded Green's function) and anticausality (advanced Green's function) is implemented in this energy-dependent Green's function [Hint: compare problem c)].

f) In the general, interacting case, the retarded Green's function contains a non-infinitesimal imaginary part in the denominator,

$$G^R_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + \mathrm{i}/(2\tau)}$$

Use the residue theorem to calculate the time-dependent Green's function by Fourier transform. How does the Green's function behave for large $(t-t')$? Give a physical interpretation for τ and try to explain why a finite τ may occur.

1 Solution

a) Apply Heisenberg equation of motion,

$$\begin{aligned} i\partial_t c_{\mathbf{k}\sigma}(t) =& - [\mathcal{H}_0, c_{\mathbf{k}\sigma}(t)]_- = -e^{i\mathcal{H}_0}[\mathcal{H}_0, c_{\mathbf{k}\sigma}(0)]_- e^{-i\mathcal{H}_0 t} && (4)\\ =& - e^{i\mathcal{H}_0 t}[\sum_{\mathbf{k}'\sigma'}(\epsilon_{\mathbf{k}'} - \mu)c^\dagger_{\mathbf{k}'\sigma'}c_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}(0)]_- e^{-i\mathcal{H}_0 t} && (5)\\ =& - e^{i\mathcal{H}_0 t}\sum_{\mathbf{k}'\sigma'}(\epsilon_{\mathbf{k}'} - \mu)[c^\dagger_{\mathbf{k}'\sigma'}c_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}]_- e^{-i\mathcal{H}_0 t} && (6)\end{aligned}$$

For Fermions:

$$[c_{\mathbf{k}'\sigma'}, c^\dagger_{\mathbf{k}\sigma}]_+ = \delta_{\mathbf{k}',\mathbf{k}}\delta_{\sigma',\sigma}, \qquad [c^{(\dagger)}_{\mathbf{k}'\sigma'}, c^{(\dagger)}_{\mathbf{k}\sigma}]_+ = 0 \tag{7}$$

$$\begin{aligned} [c^\dagger_{\mathbf{k}'\sigma'}c_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}] =& c^\dagger_{\mathbf{k}'\sigma'}[c_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}]_+ - [c^\dagger_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}]_+ c_{\mathbf{k}'\sigma'} && (8)\\ =& 0 - \delta_{\mathbf{k}',\mathbf{k}}\delta_{\sigma',\sigma}c_{\mathbf{k}'\sigma'} && (9)\end{aligned}$$

$$\begin{aligned} i\partial_t c_{\mathbf{k}\sigma}(t) =& - e^{i\mathcal{H}_0 t}\sum_{\mathbf{k}'\sigma'}(\epsilon_{\mathbf{k}'} - \mu)(-\delta_{\mathbf{k}',\mathbf{k}}\delta_{\sigma',\sigma}c_{\mathbf{k}'\sigma'})e^{-i\mathcal{H}_0 t} && (10)\\ =& (\epsilon_{\mathbf{k}'} - \mu)c_{\mathbf{k}\sigma}(t) && (11)\end{aligned}$$

$$c_{\mathbf{k}\sigma}(t) = c_{\mathbf{k}\sigma}(0)e^{-i(\epsilon_{\mathbf{k}\sigma} - \mu)t}, \qquad c^\dagger_{\mathbf{k}\sigma}(t) = c^\dagger_{\mathbf{k}\sigma}(0)e^{-i(\epsilon_{\mathbf{k}\sigma} - \mu)t} \tag{12}$$

b)

Consider the anti-commutator,

$$\begin{aligned} & [c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')]_+ && (13)\\ =& [c_{\mathbf{k}\sigma}(0)e^{-i(\epsilon_{\mathbf{k}} - \mu)t}, c^\dagger_{\mathbf{k}\sigma}(0)e^{-i(\epsilon_{\mathbf{k}} - \mu)t'}]_+ && (14)\\ =& e^{-i(\epsilon_{\mathbf{k}} - \mu)(t-t')}\underbrace{[c_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}\sigma}]_+}_{1} && (15)\\ =& e^{-i(\epsilon_{\mathbf{k}} - \mu)(t-t')} && (16)\end{aligned}$$

The retarded Green's function for Fermions,

$$G^R_{\mathbf{k}\sigma}(t-t') = -i\Theta(t-t')\frac{1}{Z_G}\mathrm{Tr}\{e^{-\beta(H-\mu N)}[c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')]_+\} \tag{17}$$

$$= -i\Theta(t-t')e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')}\underbrace{\frac{1}{Z_G}\mathrm{Tr}\{e^{-\beta(H-\mu N)}\}}_{1} \tag{18}$$

$$= -i\Theta(t-t')e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')} \tag{19}$$

in above we used $\mathrm{Tr} = \sum_n \langle n| \dots |n\rangle$ and $Z_G = \mathrm{Tr}\{e^{-\beta(H-\mu N)}\}$.

c)

Fourier transform $t \to \omega$, cause of translational invariance of time, without loss any generality we can set $t' = 0$,

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} dt\, G^{(0)R}_{\mathbf{k}\sigma}(t)e^{i\omega t} = \int dt\, (-i\Theta(t)e^{-i(\epsilon_{\mathbf{k}}-\mu-\omega)t}) \tag{20}$$

$$= \int_0^{\infty} dt\, (-ie^{-i(\epsilon_{\mathbf{k}}-\mu-\omega)t}) \tag{21}$$

need $e^{-\eta t}, \eta > 0$ where $\lim_{t\to\infty} e^{-\eta t} = 0$,

$$= \int_0^{\infty} dt\, (-ie^{-i(\epsilon_{\mathbf{k}}-\mu-\omega-i\eta)t}) \tag{22}$$

$$= -i\frac{1}{-i(\epsilon_{\mathbf{k}-\mu-\omega-i\eta})}\underbrace{[e^{-i(\epsilon_{\mathbf{k}}-\mu-\omega-i\eta)t}]_0^\infty}_{0-1} \tag{23}$$

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega-(\epsilon_{\mathbf{k}}-\mu)+i\eta} \tag{24}$$

To get the advanced Green's function, one needs to make the integral $\int_{-\infty}^0$ converge. This just result in a $-i\eta$ term in the denominator,

$$G^{(0)A}_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega-(\epsilon_{\mathbf{k}}-\mu)-i\eta} \tag{25}$$

d)

A quick way is just use the result from (b), here we use $\mathcal{H}'_0$ to denote the Hamiltonian in the first quantization from, $d_t = d/dt$,

$$(id_t - \mathcal{H}'_0)G^{(0)R}_{\mathbf{k}\sigma}(t-t') \tag{26}$$

$$=(id_t - (\epsilon_{\mathbf{k}}-\mu))(-i\Theta(t-t')e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')}) \tag{27}$$

$$=(\delta(t-t') - i\Theta(t-t')(\epsilon_{\mathbf{k}}-\mu) + i\Theta(t-t')(\epsilon_{\mathbf{k}}-\mu))e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')} \tag{28}$$

$$=\delta(t-t')e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')} \tag{29}$$

$$=\delta(t-t') \tag{30}$$

Here we used $d_t\Theta(t-t') = \delta(t-t')$.

e)

As we know from the exercise 1, the Fourier transform is given by

$$f(\omega) = \mathcal{F}(f(t)) = \int dt' f(t') e^{i\omega t'} \tag{31}$$

and a function h defined as

$$h(t) = \int dt' f(t') g(t-t'), \tag{32}$$

we have

$$h(\omega) = f(\omega) g(\omega). \tag{33}$$

And a very useful identity,

$$\mathcal{F}(df(t)/dt) = -i\omega f(\omega). \tag{34}$$

The above identity used the integration by parts, i.e.

$$\mathcal{F}(d_t f(t)) = \int dt' d_{t'} f(t') e^{i\omega t'} = [f(t')e^{i\omega t'}]_{-\infty}^{\infty} - \int dt' f(t') d_{t'} e^{i\omega t'} \tag{35}$$

In order to make the first term on the right hand side converge, for a function like G^R which is only non-vanishing if $t' > 0$, one can do substitution $\omega \to \omega + i\eta$. Then,

$$\begin{aligned}
&\mathcal{F}\left((id_t - \mathcal{H}_0')G^{(0)R}_{\mathbf{k}\sigma}(t-t')\right) = \mathcal{F}(\delta(t-t')) \\
&\Rightarrow \\
&(\omega + i\eta - (\epsilon_{\mathbf{k}} - \mu))G^{(0)R}_{\mathbf{k}\sigma}(\omega) = 1 \\
&\Rightarrow \\
&G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + i\eta}
\end{aligned} \tag{36}$$

f)

Just Fourier transform back to time space.

$$\begin{aligned}
G^R_{\mathbf{k}\sigma}(t-t') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} G^R_{\mathbf{k}\sigma}(\omega) && (37)\\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + i/2\tau} && (38)\\
&= \frac{1}{2\pi}(2\pi i\, e^{-i(\epsilon_{\mathbf{k}}-\mu) + i/2\tau)(t-t')}) && (39)\\
&= i\, e^{-i(\epsilon_{\mathbf{k}}-\mu)(t-t')} e^{-(1/2\tau)(t-t')} && (40)
\end{aligned}$$

One can see if $t \to \infty$ the damping term $e^{-(1/2\tau)(t-t')} \to 0$, this corresponds to a quasi-particle with a finite life time τ. In interacting systems the collective excitation can be damped by interacting with the system.