Universität Bonn
Physikalisches Institut
Prof. Dr. Johann Kroha, Tim Lappe, Haixin Qiu
http://www.kroha.uni-bonn.de/teaching

25 January, 2017

Special Topics in Condensed Matter Theory
Winter term 2016/17

Exercise 8

(Solutions due on 1 February, 2017)

1. Mean field theory: infinite-range Ising model (20 points)

The Hamiltonian of the ferromagnetic Ising model in an external magnetic field B is given by,

$$H = -J\frac{1}{N}\sum_{i,j}\sigma_i\sigma_j - \mu B\sum_i \sigma_i\,,$$

where $\sigma_i = \pm 1/2$ denotes the $z-$component of the spin at lattice site i. We consider a spatially infinite-range ferromagnetic interaction J, that is, the sums $\sum_{i,j}$ run unrestrictedly over all N lattice sites.

a) Show that, in the limit of $n \gg 1$, this Hamiltonian is equivalent to the following Hamiltonian of independent spins in an effective magnetic field,

$$H_{MF} = -\mu_B(B_{MF} + B)\sum_i \sigma_i$$

and express the B_{MF} in terms of the average spin per site, $\langle\sigma_i\rangle$.

b) Calculate the canonical partition function Z and the free energy F for this system.

c) Derive a selfconsistent equation for the mean field B_{MF} by minimizing the free energy. How is this equation related to the expression of problem a) for B_{MF} in terms of $\langle\sigma_i\rangle$?

d) Solve the selfconsistent equation graphically for $B = 0$ and sketch the magnetization $M(T)$ (order parameter) as a function of temperature T. Determine the critical temperature T_c in terms of J.

e) Now solve the selfconsistent equation graphically for $B > 0$ and sketch $M(T)$.

f) Let $\tau = (T - T_c)/T_c$ the reduced temperature. Calculate the magnetization $M(\tau)$ analytically in the vicinity of $T = T_c$ by expanding for small order parameter values. Why is such an expansion possible?

g) Calculate the magnetic susceptibility $\chi(\tau)$ in the vicinity of $T = T_c$. What is the critical exponent of the susceptibility?