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Special Topics in Condensed Matter Theory
Winter term 2016/17

Exercise 6

(Solutions due on January 13, 2017)

In the lecture we learned the Feynman-diagram technique for imaginary time Green functions or Matsubara Green functions. This corresponds to finite-temperature quantum systems, e.g. the electron gas. Here we apply it and consider an important second-order diagram for particle-particle interaction.

6.1 Feynman Diagrams of Imaginary Time Green Functions (10 points)

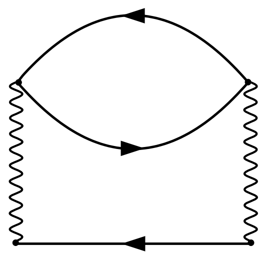
The Hamiltonian of a many-electron system reads,

$$H = \sum_{\mathbf{p}\sigma} (\epsilon_p - \mu) c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} + \sum_{\sigma\sigma'} \int d^3 r_1 \int d^3 r_2 c^\dagger_{\mathbf{r}_1\sigma} c_{\mathbf{r}_2\sigma} W(|\mathbf{r}_1 - \mathbf{r}_2|) c^\dagger_{\mathbf{r}_{2\sigma'}} c_{\mathbf{r}_2\sigma'}, \quad (1)$$

where we consider a continuous system with single-particle energy $\epsilon_\mathbf{p}$ and chemical potential μ, with a particle-particle interaction strength $W(|\mathbf{r}_1 - \mathbf{r}_2|)$.

a) Consider the one-particle (two-point) Matsubara Green's function for Fermions $G(\mathbf{k}, ik_n)$, where $\mathbf{k}$ denotes the momentum and k_n denotes the Matsubara frequency. Write down Fourier space Feynman rules for particle-particle interaction, i.e. for the Hamiltonian in Eq. (1).

b) Draw all Feynman diagrams up to second order in interaction strength $W(\mathbf{q})$. Figure out which parts are belong to the self-energy $\Sigma(\mathbf{k}, ik_n)$. Label the diagram below with appropriate momentum and Matsubara Frequencies, it is usually called *pair-bubble self energy*.



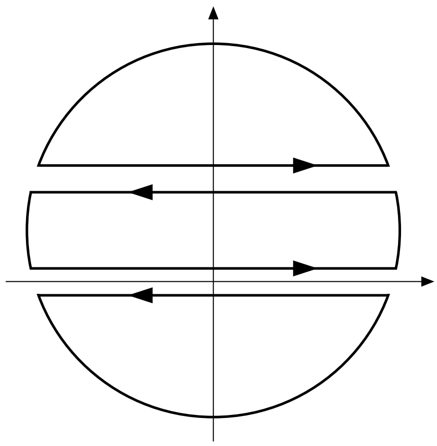
c) Use Fourier space Feynman rules to write down the explicit formula of the *pair-bubble self energy*. One should reach something like,

$$\Sigma^P_\sigma(\mathbf{k}, ik_n) = \frac{1}{\beta} \sum_{iq_n} \int \frac{d^3 q}{(2\pi)^3} W(\mathbf{q})^2 \Pi^0(\mathbf{q}, iq_n) G^0_\sigma(\mathbf{k} - \mathbf{q}, ik_n - iq_n). \quad (2)$$

Draw $\Pi^0(\mathbf{q}, iq_n)$ into the Feynman diagram. This diagram corresponds to the so-called *pair-bubble* and also appears when one calculates the retarded density-density correlation function $\chi^R(\mathbf{q}, \omega)$ (this may remind you of problem sheet 1).

<u>6.2 Lindhard function</u> (10 points)

a) Perform the Matsubara summation in $\Pi^0(\mathbf{q}, iq_n)$. Remind yourself that iq_n is a bosonic Matsubara frequency (why?). Before doing so, just finish the graph below, label the axes, place the poles and argue where are the branch cuts. [Hint: The free Green's function is $G^0_\sigma(\mathbf{p}, ip_n) = 1/(ip_n - (\epsilon_\mathbf{p} - \mu)]$



b) Now we analytically continue the $\Pi^0(\mathbf{q}, iq_n)$ to its retarded component $\Pi^R(\mathbf{q}, \omega)$ ($\sim \chi^R(\mathbf{q}, \omega)$). Write down the relation between $\Pi^0(\mathbf{q}, iq_n)$ and $\Pi^R(\mathbf{q}, \omega)$. Show that for $\omega = 0$ (the static case), we have

$$\begin{aligned}\Pi^R(\mathbf{q}, 0) &= 2 \int \frac{dk}{(2\pi)^3} \frac{f(\epsilon_{\mathbf{k}+\mathbf{q}} - \mu) - f(\epsilon_\mathbf{k} - \mu)}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_\mathbf{k}} \\ &\overset{T \to 0}{=} 2 \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(\mu - \epsilon_{\mathbf{k}+\mathbf{q}/2}) - \Theta(\mu - \epsilon_{\mathbf{k}-\mathbf{q}/2})}{\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2}}.\end{aligned}$$

c) For small momentum transfer $|\mathbf{q}|$, the analytic solution of $\Pi^R(\mathbf{q}, 0)$ is,

$$\Pi^R(\mathbf{q}, 0) = \begin{cases} \dfrac{m}{\pi \epsilon_F} \dfrac{1}{x} \ln \left| \dfrac{1-x}{1+x} \right| & d = 1 \\ -q_{TF}^2 \dfrac{1}{2} \left(1 + \dfrac{1-x^2}{2x} \ln \left| \dfrac{1-x}{1+x} \right| \right) & d = 3 \end{cases} \tag{3}$$

where $x = \mathbf{q}/2k_F$, k_F is the Fermi momentum, and ϵ_F is the Fermi energy. In $d = 3$ dimensions, $q_{TF} = \sqrt{4e^2 m \epsilon_F / \pi}$ is the so-called Thomas-Fermi wave number. $\Pi^R(\mathbf{q}, 0)$ is called the *Lindhard function*. Sketch "$\Pi^R(\mathbf{q}, 0)$ vs. x" for $1d$ and $3d$ solutions.