Universität Bonn 18 November, 2016
Physikalisches Institut
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http://www.kroha.uni-bonn.de/teaching

Special Topics in Condensed Matter Theory
Winter term 2016/17

Exercise 3

(Solutions due on 25 November, 2016)

1. Green's functions: general properties (10 points)

On the previous exercise, you calculated explicitely the retarded Green's function G for the special case of a diagonal Hamiltonian. However, in most cases the system is more complex, e.g., in the presence of interactions. Nevertheless some general properties of G will always hold. Some of them will be discussed in this exercise.

a) Normalization.
The definition of the spectral function $A_{\mathbf{k}\sigma}(\omega)$ is (see lecture),

$$A_{\mathbf{k}\sigma}(\omega) \;=\; \frac{1}{Z_G}\sum_{n,m} |\langle n|c_{\mathbf{k}\sigma}|m\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right)\, \delta(\omega + E_n - E_m), \qquad (1)$$

Using the definition, show that $A_{\mathbf{k}\sigma}(\omega)$ is normalized,

$$\int_{-\infty}^{\infty} d\omega\, A_{\mathbf{k}\sigma}(\omega) \;=\; 1. \qquad (2)$$

Use the spectral representation of $G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega)$,

$$G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) \;=\; \int_{-\infty}^{\infty} d\omega'\, \frac{A_{\mathbf{k}\sigma}(\omega')}{\omega - \omega' + \mathrm{i}0^+}, \qquad (3)$$

to find the relation between $\mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega)$ and $A_{\mathbf{k}\sigma}(\omega)$. Calculate

$$\int_{-\infty}^{\infty} d\omega\; \mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega). \qquad (4)$$

b) Asymptotic behavior.
You can assume that $A_{\mathbf{k}\sigma}(\omega) = 0$, if $|\omega| > D$ finite D, $0 < D < \infty$. (finite band width D of lattice systems). Show

$$\lim_{\omega\to\pm\infty} \omega \cdot G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) \;=\; 1, \qquad (5)$$

i.e., $G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) \approx 1/\omega$ for large energies.

2. Density of states (10 points)

The density of states $N^{(d)}(\omega)$ is an important characteristic of a many-body system. In particular, it depends on the dimensionality of the system. It is defined as,

$$N^{(d)}(\omega) = \int d^d k \, A_{\mathbf{k}\sigma}(\omega) \ , \tag{6}$$

where $d = 1,\ 2,\ 3$ is the spatial dimensionality. The non-interacting, momentum-dependent Green's function is [c.f. exercise 2, Eq. (3)],

$$G_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\varepsilon_{\mathbf{k}} - \mu) + i\eta} \ . \tag{7}$$

For a gas of free electrons with energy $\varepsilon_{\mathbf{k}} = (\hbar k)^2/2m$, calculate the density of states $N^{(d)}(\omega)$ for $d = 1,\ 2,\ 3$.

Hint: Use the relation between spectral function $A_{\mathbf{k}\sigma}(\omega)$ and $G_{\mathbf{k}\sigma}(\omega)$. Transform the momentum integrals in d dimensions into an integral over the energy ε by an appropriate substitution.

Sketch the density of states $N^{(d)}(\omega)$ as a function of the particle energy ω for $d = 1,\ 2,\ 3$, respectively.