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Advanced Theoretical Condensed Matter Physics
Exercise 5

(Submission date: 03.07.19, discussion: 05.07.19)

<u>5.1 Renormalization group (RG) for the anisotropic Kondo model</u> (3+3+5+4+5=20 points)

The Hamiltonian of the anisotropic Kondo model reads,

$$H = \sum_{\boldsymbol{p}\sigma} (\varepsilon_{\boldsymbol{p}} - \mu) c^{\dagger}_{\boldsymbol{p}\sigma} c_{\boldsymbol{p}\sigma} + J_{||} \hat{S}_z \, \hat{\sigma}_z + J_{\perp} \left(\hat{S}^{+} \hat{\sigma}^{-} + \hat{S}^{-} \hat{\sigma}^{+} \right) \ , \tag{1}$$

where $\hat{S}_i$, $i = x, y, z$, are the spin operators in impurity spin space, and the conduction electron spin operators are $\hat{\sigma}_i = \sum_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'} c^{\dagger}_{\boldsymbol{p}\sigma} \sigma_i c_{\boldsymbol{p}'\sigma'}$, $i = x, y, z$, with the Pauli matrices σ_i.
In the lecture the perturbative RG equations for the dimensionless coupling constants $g_{||} = N(0) J_{||}$ and $g_{\perp} = N(0) J_{\perp}$ (to one-loop order) were derived as,

$$\frac{d g_{||}}{d \ln D} = -2 \, {g_{\perp}}^2 \tag{2}$$

$$\frac{d g_{\perp}}{d \ln D} = -2 \, g_{||} \, g_{\perp} \ , \tag{3}$$

where the density of states is assumed to be flat and takes the value $N(0) = 1/(2D_0)$ at the Fermi surface, D_0 being half the bare conduction bandwidth (cutoff). We now want to solve these equations for arbitrary (anisotropic) bare couplings $g_{||\,0}$, $g_{\perp\,0}$.

a) From the above RG equations, derive a single differential equation in terms of $dg_{||}/dg_{\perp}$ which relates $g_{||}$ and $g_{\perp}$ to each other, and which does not explicitly depend on the cutoff D.

b) Solve the differential equation obtained in problem a) by separation of variables. What type of curves are the solutions $g_{\perp}(g_{||})$ or $g_{||}(g_{\perp})$?

c) Draw the family of $g_{\perp}(g_{||})$ trajectories in a $g_{\perp}$ vs. $g_{||}$ diagram. Note that $g_{\perp}$ and $g_{||}$ may be positive or negative.

d) Assign to each trajectory an arrow which marks the flow direction of the coupling constants along that trajectory, using the RG equations (2), (3). Using these flow directions, identify the three different types of low-energy RG fixed points. (Note: $g \to \infty$ is also a fixed point in the RG sense.) Which of these fixed points are stable, which ones are unstable? Discuss which physical ground state each fixed point corresponds to.

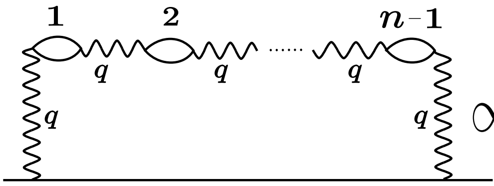
e) Knowing $g_{\perp}(g_{||})$ from problem b), now determine the "running" coupling constant $g_{||}(D)$ from the RG equation (2) as a function of the cutoff D. Determine $g_{\perp}(D)$ as well.

5.2 Screened Coulomb potential (6+3+3+3+4+4+3= 26 points)

In a dense system of electrons (electron sea) the electrostatic potential effectively created by one electron is modified (renormalized) as compared to the bare Coulomb potential, because all electrons of the system interact and, hence, are repelled from that electron, so that a positive charge cloud is created around the (negatively charged) electron. This charge cloud weakens the bare Coulomb potential and reduces the spatial extent of the interaction.
We will aim at capturing this effect by partially summing up the important classes of diagrams in the self energy. For this we will look at the scattering amplitude for the Coulomb interaction in a similar way as has been done for the Kondo problem in the lecture.

a) First we must think about, which class of diagrams will be the most important in the situation at hand. The diagrams can be classified in terms of their scaling in density and in terms of their low-momentum divergences ("infrared divergence"). The measure for the density scaling is the power of Fermi momentum, which is attached to the diagram. For the infrared scaling the least restricted integral is important.
Draw all diagrams contributing to the self energy up to second loop order and two contributions of third loop order. Classify them in terms of the Fermi momentum and the degree of divergence.
Hint: Each Matsubara sum will give a factor of k_f^2.

For the high density case the most important diagrams are of the type drawn here. This class of diagrams can be included by considering a self-consistent Dyson-like equation for the interaction line (scattering amplitude).



$$\propto k_f^{-n} \int \frac{d^3q}{q^{2n+4}}$$

This new equation describes the change of the Coulomb force due to the interaction between many electrons. Inside the electron self energy the new interaction lines should be used. This changes the diagrams appearing in the self energy. Here one must watch out not to overcount diagrams, which might already be generated by the suplemented equation.

b) Write down a Dyson equation for the interaction line which includes all corrections of the type drawn in the figure. Solve it for the new interaction line. Here sheet 3 and 4 might be helpful.

In the following we just keep the Fock diagram in the electron self energy, but with the full interaction line.

c) Evaluate the Matsubara sum. Note that the new interaction is frequency dependent and therefore carries a bosonic Matsubara frequency.

d) Take the imaginary part of the self energy and perform with this the remaining frequency integral. It is now useful to express the imaginary part of the interaction line in terms of the imaginary part of the general Lindhard function given on the last sheet. Note that it contains a delta function.

We are just interested in the behavior close to the pole, therefore set the external frequency of the self energy equal to the dispersion at the respective momenta. Here one obtains:

$$\mathrm{Im}\Sigma^R(k,\epsilon_k) = \pi \sum_{q,p} [b(\epsilon_{k+q}-\epsilon_k) + f(\epsilon_{k+q})][f(\epsilon_{p+q}) - f(\epsilon_p)] \left| \frac{V(q)}{1 - V(q)\chi^R(q,\epsilon_{k+q}-\epsilon_k)} \right|^2$$
$$\times\, \delta(\epsilon_p - \epsilon_k + \epsilon_{k+q} - \epsilon_{p+q})$$

To proceed further the distribution functions in the expression should be analysed. In the following is $f(\epsilon)$ the Fermi function and $b(\epsilon)$ the Bose function.

e) First show that the following two equalities hold:

$$b(\epsilon_1 - \epsilon_2)\,[f(\epsilon_2) - f(\epsilon_1)] = f(\epsilon_1)[1 - f(\epsilon_2)] = f(\epsilon_1)f(-\epsilon_2)$$
$$f(\epsilon_1)f(-\epsilon_2) - f(\epsilon_2)f(-\epsilon_1) = f(\epsilon_1) - f(\epsilon_2)$$

f) With this show that the distribution functions in the self energy can be written as

$$f(\epsilon_{p+q})f(-\epsilon_{k+q})f(-\epsilon_p) + f(-\epsilon_{p+q})f(\epsilon_{k+q})f(\epsilon_p)\,.$$

g) Argue that for k close above the Fermi surface the self energy scales as

$$\mathrm{Im}\Sigma^R(k, \epsilon_k) \propto \epsilon_k^2$$

and therefore the quasi-particle lifetime $1/\tau \propto \epsilon_k^2$.