Universität Bonn
Physikalisches Institut
Prof. Dr. J. Kroha, M. Kajan, M. Lenk, M. Turaev

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Advanced Theoretical Condensed Matter Physics
Exercise 4

(Submission date: 19.06.19, discussion: 21.06.19)

4.1 First corrections of Fermi liquid (3+5+2+6+4=20 points)

We want to systematically include interaction between electrons in our model of fermions. The interaction we consider has the form

$$H_I = \frac{1}{2}\int d^3x \int d^3y \Psi^\dagger(x)\Psi^\dagger(y)V(x-y)\Psi(y)\Psi(x)\,. \tag{1}$$

a) Show that interaction in momentum space is given by

$$H_I = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\; c^\dagger_{p-q}c^\dagger_{k+q}V_q c_k c_p\,.$$

The two diagrams contributing to first order you have already seen in the lecture, the Hartree and the Fock diagram.

b) Draw the two first-order self-energy diagrams and attach momentum and frequency labels to the Green's function and interaction lines. Perform the Matsubara sums by including the appropriate convergence factor and obtain

$$\Sigma(\omega,k) = \int \frac{d^3p}{(2\pi)^3}\left(V_{q=0} - V_{k-p}\right)\int \frac{d\epsilon}{2\pi} f(\epsilon)A_p(\epsilon)$$

c) The first term is divergent for a Coulomb interaction $V(q)$ between the electrons. Why is that the case and how is this divergence canceled in physical systems?

The first-order self-energy does not depend on frequency and, thus, has no imaginary part (why?). Therefore, it will only modify ("renormalize") the dispersion relation ϵ_k, such that the retarded Green's function is given by

$$G_k^R(\omega) = \frac{1}{\omega - \epsilon_k^* + i\eta} \qquad \text{with} \qquad \epsilon_k^* = \epsilon_k + \Sigma_k\,. \tag{2}$$

d) Calculate the effective mass m^*, defined by $\frac{\vec{k}-\vec{p}_f}{m^*} = \vec{\nabla}_k \epsilon_k^*$ for momenta close to the Fermi momentum p_F , in the zero temperature limit.
Hint: The Fermi function is proportional to a Heaviside step function in this limit. Note also that $\int dp^3\vec{p} = \hat{k}\int dp^3\vec{p}\cdot\hat{k}$, with $\hat{k}\cdot\hat{k} = 1$.

In Fermi liquid theory, the result is expressed in terms of the Landau parameter F_1^s, with $D^* = \frac{m^* p_F^*}{\pi^2}$ the density of states of the interacting Fermi system at the Fermi level,

$$\frac{m^*}{m} = 1 + F_1^s \quad \text{with} \quad F_1^s = -D^*\int \frac{d\Omega}{4\pi}\frac{V_{k-p}}{2}\cos(\theta)\,.$$

e) Show that the same self-energy as in b) is obtained by ordering the operators in the interaction (1) to products of the form $c^\dagger c$ (not possible "−" signs) and replacing it by its expectation value $\langle c^\dagger c\rangle$ (mean-field approx.). Why is this a self-consistent approximation?

4.2 Density Correlation II (4+3+11+3+11+3=35 points)

We continue here the calculation of the density response function started on the last exercise sheet.

a) Evaluate the Matsubara sum of the expression below and use that the spectral functions appearing here are proportional to a delta function.

$$\chi(q, i\omega) = \frac{1}{\beta} \sum_{i\nu} \int \frac{d^3k}{(2\pi)^3} G(k+q, i\omega + i\nu) G(k, i\nu) \tag{3}$$

The results is given by the Lindhard function

$$\chi^R(q,\omega) = \int \frac{d^3k}{(2\pi)^3} \frac{f(\epsilon_k) - f(\epsilon_{k+q})}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta} .$$

We want to compute now the momentum integral at zero temperature, where the Fermi function reduces to Heaviside function up to the Fermi energy.

b) To evaluate the integral we can shift $\vec{k} \to \vec{k} - \vec{q}/2$ or $\vec{k} \to \vec{k} - \vec{q}$. Choose spherical coordinates with the angle θ between $\vec{k}$ and $\vec{q}$.

c) Solve the remaining integrals. This can be done by either using $\frac{1}{x+i\eta} = P\frac{1}{x} - i\pi\delta(x)$ or by treating the appearing logarithms as complex functions and in the end determining real and imaginary parts through the sign of the argument.
Hint:

$$\int_0^b dx\, x \log\frac{a+x}{a-x} = \left[\frac{1}{2}x^2 \log\frac{a+x}{a-x}\right]_0^b - \frac{1}{2}\int_0^b dx\, x^2 \left(\frac{1}{a+x} + \frac{1}{a-x}\right)$$

d) Separate the expression into real and imaginary part. Where is the imaginary part non-zero and what does it correspond to?

We now want to look at the real space behavior of the density correlation. The integrals appearing in the Fourier transformation are far from trivial for $\omega \neq 0$.

e) Perform the Fourier transformation to real space of

$$\chi^R(q, \omega = 0) = \frac{mk_f}{(2\pi)^2}\left[1 + \frac{1}{\tilde{q}}\left[\left(1 - (\tilde{q}/2)^2\right)\log\left|\frac{\tilde{q}/2+1}{\tilde{q}/2-1}\right|\right]\right] \quad \text{with} \quad \tilde{q} = q/k_f .$$

Hint: Perform the integrals over the angles first by using the fact that χ only depend on the absolute value of the momenta. The expression obtained can be partially integrated a few times and simplified with the addition theorem for sin and cos. Note that $\int_0^\infty dx \frac{\sin(x)}{x} = \frac{\pi}{2}$

f) What is the characteristic oscillation frequency and long-range behavior?