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Advanced Theoretical Condensed Matter Physics
Exercise 3

(Submission date: 29.05.19, discussion: 31.05.19)

On this problem sheet we practise the use of the Matsubara technique.

2.1 Particle density and momentum distribution of a Fermi gas (2+8=10 points)

For clarity we give again the conventions for the Fourier transformations in momentum and in Matsubara space:

$$\Psi(\boldsymbol{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} c_k \ , \qquad B(\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n\tau} B(i\omega_n) \ , \qquad B(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau} B(\tau) \tag{1}$$

The Matsubara (or thermal) Green's function is defined in position or in momentum space, respectively, as (with $x = (\boldsymbol{r}, \tau)$, $\tau' = 0$),

$$G(x, x') = -\langle \hat{T}[\Psi(x)\Psi^\dagger(x')]\rangle \ , \qquad G(\boldsymbol{k}, \tau) = -\langle \hat{T}[c_{\boldsymbol{k}}(\tau)c^\dagger_{\boldsymbol{k}}(0)]\rangle \ . \tag{2}$$

a) Express the expectation value of the particle density $\langle n(\boldsymbol{r})\rangle = \langle \Psi^\dagger(x)\Psi(x)\rangle$ in terms of $G(x, x)$ as well as in terms of $G(\boldsymbol{k}, \tau)$.

b) Express the momentum distribution (i.e., avg. occupation number of a $\boldsymbol{k}$-state) $\langle n(\boldsymbol{k})\rangle = \langle c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}}\rangle$ in terms of $G(\boldsymbol{k}, \tau)$. Use the Fourier decomposition in Matsubara space (second equality of (1)) and perform the Matsubara sum by contour integration to show that

$$\langle n(\boldsymbol{k})\rangle = \int d^3x\, e^{-ixq}\langle n(x)\rangle = \int \frac{d\varepsilon}{2\pi} f(\varepsilon) A_{\boldsymbol{k}}(\varepsilon) \,,$$

where $A_{\boldsymbol{k}}(\varepsilon)$ is the spectral function.

2.2 Density Correlations in a Fermi gas (7+3+4+3+8=25 points)

Due to the Pauli principle, two electrons of equal spin cannot be simultaneously at the same point in space. Because the electron wave-functions are continuous, this effect of Fermi statistics induces spatially extended, repulsive density correlations in a Fermi gas even without interactions, so-called statistical correlations. To describe this effect quantitatively, we calculate the density response function in three dimensions for equal spins σ,

$$\chi^R(\boldsymbol{r} - \boldsymbol{r}', t) = -i\Theta(t)\langle[\hat{n}_\sigma(\boldsymbol{r}, t), n_\sigma(\boldsymbol{r}', 0)]\rangle \ , \qquad \hat{n}_\sigma(\boldsymbol{r}, t) = \Psi^\dagger_\sigma(x)\Psi_\sigma(x) \tag{3}$$

The spin index σ will be dropped in the following.

a) Show that the imaginary-time ordered correlation function $\chi(\boldsymbol{r}-\boldsymbol{r}', i\omega)$, defined through the imaginary-time Fourier transformation of

$$\chi(\boldsymbol{r}-\boldsymbol{r}', \tau) = -\langle \hat{T}(\hat{n}(\boldsymbol{r}, \tau)\hat{n}(\boldsymbol{r}', 0))\rangle \ , \tag{4}$$

is related to the retarded correlation function $\chi^R(\boldsymbol{r}-\boldsymbol{r}', \omega)$ by analytic continuation with $i\omega \to \omega + i\eta$. To this end, use the energy eigenbasis and perform the Fourier transformation. Hint: The density operator is of bosonic nature and therefore periodic w.r.t. imaginary time with period inverse temperature β.

Thus, we have related the retarded correlation function to the thermal correlation function which can now be analyzed using Wick's theorem.

b) Momentum representation. Use the momentum representation of the density operators, $\hat{n}(\boldsymbol{q}, \tau)$, to show that

$$\chi(\boldsymbol{q}, \tau) = \int d^3r\, d^3r' e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}\chi(\boldsymbol{r}-\boldsymbol{r}', \tau) = -\langle \hat{T}(\hat{n}(\boldsymbol{q}, \tau)\hat{n}(-\boldsymbol{q}, 0))\rangle \ . \tag{5}$$

c) Use the representation of $\hat{n}(\boldsymbol{q}, \tau)$ in terms of creation and annihilation operators and do the Wick contractions. Draw the corresponding two Feynman diagrams, determine the sign of each diagram and label all Green's function lines with momenta and frequencies. Why is a "disconnected" diagram contributing to $\chi(\boldsymbol{q}, \tau)$? Why is this diagram just a space- and time-independent constant C? Write down C in terms of the particle density $\langle \hat{n} \rangle$.
Hint: The Green's function is diagonal in momentum space.

d) Show that

$$\chi(\boldsymbol{q}, \tau) = \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}+\boldsymbol{q}, \tau)G(\boldsymbol{k}, -\tau) \ + \ C\,\delta^3(\boldsymbol{q}) \ . \tag{6}$$

e) We now consider the local (in space) response function $\chi(\boldsymbol{r}=0, \tau)$ which can be obtained by summing equation (6) over all momenta $\boldsymbol{q}$. Show that

$$\chi(\boldsymbol{r}=0, \tau) = \int \frac{d^3q}{(2\pi)^3}\, \chi(q, \tau) = G(\boldsymbol{r}=0, \tau)G(\boldsymbol{r}=0, -\tau)\,. \tag{7}$$

This is a great simplification, because in equation (7) no momentum convolution occurs.

f) Transform equation (7) from imaginary time to Matsubara frequency and perform the Matsubara sum by contour integration. Note that the Green's functions have a branch cut at the real frequency axis. Perform the analytic continuation of the result to the real axis, $i\omega_n \to \omega + i\eta$, to obtain the causal response function $\chi^R(\boldsymbol{r}-\boldsymbol{r}'=0, t)$.
Hint: $\chi(i\omega_n)$ is of bosonic nature and the Fermi function $f(z)$ is periodic w.r.t. bosonic Matsubara frequencies $f(z) = f(z + i\omega_n)$.