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Advanced Theoretical Condensed Matter Physics
Exercise 2

(Submission date: 15.05.19, discussion: 17.05.19)

2.1 Green's functions: general properties (5+3+4+3=15 points)

On the previous exercise sheet, you calculated explicitly the retarded Green's function G for the special case of a diagonal Hamiltonian. However, in most cases the system is more complex, e.g. in the presence of interactions. Nevertheless, some general properties of G always hold due t oits analyticity. Consider the retarded Green's function for bosons and fermions,

$$G_k^R(t,t') = -\mathrm{i}\Theta(t-t')\frac{1}{Z_G}\mathrm{tr}\left\{\mathrm{e}^{-\beta\mathcal{H}}\left[c_k(t), c_k^\dagger(t')\right]_\pm\right\} \tag{1}$$

where the upper sign is for fermions, the lower one for boson.

a) Show that the spectral function $A_k(\omega) = -2\,\mathrm{Im}(G_k^R(\omega))$ can be written as:

$$A_k(\omega) = \frac{2\pi}{Z_G}\sum_{n,m}|\langle n|c_k|m\rangle|^2\left(e^{-\beta E_n} \pm e^{-\beta E_m}\right)\delta(\omega + E_n - E_m). \tag{2}$$

In order to show this, proceed as shown in the lecture by writing out the trace in Eq. (1) in the eigenbasis of the Hamiltonian $\mathcal{H}$ and perform the Fourier transform.

b) Using Eq. (2), show that the spectral function is normalized,

$$\int\frac{\mathrm{d}\omega}{2\pi}A_k(\omega) = 1\ .$$

c) Show that $A_k(\omega) \geq 0$ for fermions and $\omega A_k(\omega) \geq 0$ for bosons.

d) Asymptotic behavior: From the previous tasks it is clear that $A_k(\omega)$ will decay to zero at large frequencies. Assume that $A_k(\omega)$ is non-zero only in a region $|\omega| < D$ and use the spectral representation of $G_k^R(\omega)$

$$G_k^R(\omega) = \lim_{\eta\to 0}\int\frac{\mathrm{d}\omega'}{2\pi}\frac{A_k(\omega')}{\omega - \omega' + i\eta}$$

to show that:

$$\lim_{\omega\to\infty}\omega\, G_k^R(\omega) = 1$$

2.2 Sommerfeld expansion (2+7+6=15 points)

We are often dealing with fermions at low temperatures compared to the Fermi energy. The Sommerfeld expansion is a useful tool to derive expressions for small temperatures $1/\beta$.
The quantities one is interested in are typically of the form

$$I = \int_{-\infty}^{\infty} \mathrm{d}\varepsilon f(\varepsilon)\; H(\varepsilon), \qquad \text{with} \quad f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \quad \text{the Fermi function.} \tag{3}$$

Here, $H(\varepsilon)$ is a function which vanishes for $\varepsilon \to -\infty$ and grows not faster than polynomial as $\varepsilon \to \infty$. μ is the chemical potential.

a) To derive the Sommerfeld expansion, perform a partial integration of Eq. (3) and show

$$I = -\int_{-\infty}^{\infty} \mathrm{d}\varepsilon f'(\varepsilon) K(\varepsilon) \quad \text{with} \quad K(\varepsilon) = \int_{-\infty}^{\varepsilon} \mathrm{d}\varepsilon' H(\varepsilon')\,.$$

Here $f'(\varepsilon)$ is the derivative of the Fermi function.
Sketch $f'(\varepsilon)$ and give a measure for the characteristic width.

b) Due to the strong localization of $f'(\varepsilon)$ at the chemical potential, K can be expanded in powers of εT around it. Show that

$$I = \int_{-\infty}^{\mu} \mathrm{d}\varepsilon H(\varepsilon) + \sum_{n=1}^{\infty} T^{2n} a_n H^{(2n-1)}(\mu),$$

where $H^{(2n-1)}(\mu)$ denotes the $(2n-1)^{\text{st}}$ derivative of H evaluated at μ, and $a_n = \frac{1}{(2n)!}\int_{-\infty}^{\infty} \mathrm{d}x \frac{x^{2n} e^x}{(e^x+1)^2}$.

The lowest order correction then takes the form

$$I = \int_{-\infty}^{\mu} \mathrm{d}\varepsilon H(\varepsilon) + \frac{\pi^2}{6} T^2 H'(\mu)\,.$$

We now want to compute the leading-order temperature dependence of the chemical potential for fixed, average particle number. To that end, we consider the integral in Eq. (3), with H replaced by the density of states $D(\varepsilon)$. The integral is then equal to the total particle number N. For a physical system, $D(\varepsilon)$ must be bounded from below.

c) When the average particle number N is fixed, μ will depend on temperature T. Find $\mu(T)$ up to and including corrections of $\mathcal{O}(T^2)$. Use that you can evaluate Eq. (3) exactly at $T = 0$. You should obtain

$$\mu = \varepsilon_F - \frac{\pi^2}{6} T^2 \frac{D'(\varepsilon_F)}{D(\varepsilon_F)}\,.$$