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Physikalisches Institut
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Advanced Theoretical Condensed Matter Physics
Summer term 2018

Exercise 6

(Solutions due on July 10, 2018, 12h)

6.1 Linear response and ferromagnetism in the Hubbard model (15 points)

We consider the Hubbard model,

$$H = -t \sum_{\langle i,j\rangle\sigma} c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_{i} \hat{n}_{i\uparrow}\hat{n}_{i\downarrow} \qquad \hat{n}_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}\ ,\ t > 0\ , \tag{1}$$

with the filling factor, i.e., the average particle number per site, $\nu = 1/N\ \sum_{i\sigma}\langle \hat{n}_{i\sigma}\rangle$. While near half filling ($nu \approx 1$) the effective spin exchange coupling is antiferromagnetic (see lecture), the Hubbard model can support a ferromagnetic phase for small filling factors, $\nu \ll 1$, if the Hubbard repulsion U is sufficiently large. The ferromagnetic instability of a sea of itinerant electrons due to short-range, repulsive electrostatic interaction is called Stoner ferromagnetism. At a ferromagnetic phase transition an infinitesimally small external magnetic field is large enough to induce a finite magnetization, i.e., the static ($\Omega = 0$), uniform ($q = 0$) spin susceptibility $\chi_{ss}(\Omega = 0, q = 0)$ diverges. Here we will explore this behavior.

a) The z-z component of the spin susceptibility (longitudinal susceptibility) is given in the time and position domain by,

$$\chi_{s_z sz}(t - t', \mathbf{r} - \mathbf{r}') = -i\Theta(t - t')\langle[s_z(t, \mathbf{r}), s_z(t', \mathbf{r}')]\rangle\ , \tag{2}$$

with the electron spin operator at site i, $s_z = \frac{1}{2}\sum_{\sigma\sigma'} c^{\dagger}_{i\sigma}\sigma^{z}_{\sigma\sigma'} c_{i\sigma'} = \frac{1}{2} c^{\dagger}_{i\uparrow} c_{i\uparrow} - c^{\dagger}_{i\downarrow} c_{i\downarrow}$. Using this definition, express the spin susceptibility in terms of the density response function, $\chi_{\rho\rho}$.

b) Consider now the spin susceptibity $\chi_{s_z s_z}(\Omega, \mathbf{q})$ and the density response function $\chi_{\rho\rho}(\Omega, \mathbf{q})$ in frequency and momentum space. Express the spin susceptibility of the interacting system as an infinite series of *chain diagrams* in terms of the Hubbard interaction U and the polarization *bubble diagram* $\Pi(\Omega, \mathbf{q})$, sum up this infinite series of terms and express it in terms of $\Pi(\Omega, \mathbf{q})$.

   Note: This summation is almost identical to the summation in ex sheet 4, problem 4.1. Itinvolves no explicit computation of diagrams.

c) Now consider the static and uniform limit of the result of b): $\Omega = 0$, $q = 0$. Show that, for a non-interacting system, the static, uniform density response at temperature $T = 0$ is given by

$$\chi^{(0)}_{\rho\rho}(\Omega = 0, \mathbf{q} = 0) = \frac{dn}{d\mu} = N^{(0)}(\varepsilon_F)\ , \tag{3}$$

where n is the average particle number per site (particle density), μ the chemical potential, and $N^{(0)}(\varepsilon_F)$ the density of states of the non-interacting system at the Fermi level.

Note: This relation can be shown using the definitions of the respective quantities from linear response theory only (without evaluation of diagrams). It is, therefore, generally valid for any non-interacting Fermi system.
Note for the proof that a change of a static, uniform electrostatic potential, applied to the system, is just a change of the chemical potential μ and that the density response function describes, by definition, the change of particle density n due to that applied potential.

d) For a non-interacting system, the (bare) density response function $\chi^{(0)}_{\rho\rho}(\Omega, \mathbf{q})$ coincides with the retarded polarization function $\Pi^R(\Omega, \mathbf{q})$ (why?). Using the results of b) and c), now write down the static, uniform spin susceptibility of the Hubbard model for $T = 0$ (valid for small filling factor) and show that it has a divergence when

$$U N^{(0)}(\varepsilon_F) = 1 \ . \tag{4}$$

Hence, this is the condition on U and $N^{(0)}(\varepsilon_F)$ for a ferromagnetic transition to occur, the so-called *Stoner criterion.*