**Universität Bonn** May 30, 2018
**Physikalisches Institut**
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http://www.kroha.uni-bonn.de/teaching

# Advanced Theoretical Condensed Matter Physics
# Summer term 2018

## Exercise 4

(Solutions due on June 12, 2018, 12h)

Two electrons interact via the Coulomb potential, $v(r) = v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = e^2/r$. However, in a dense system of electrons (electron sea) the electrostatic potential effectively created by one electron is modified (renormalized) as compared to this bare Coulomb potential, because all electrons of the system interact and, hence, are repelled from that electron, so that a positive charge cloud is created around the (negatively charged) electron. This charge cloud weakens the bare Coulomb potential and reduces the spatial extent of the interaction. This effect is called screening and will be investigated in this problem set.

### 4.1 Screened Coulomb potential (10 points)

The Hamiltonian of a many-electron system reads,

$$H = \sum_{\boldsymbol{p}\sigma} (\varepsilon_{\boldsymbol{p}} - \mu) c^\dagger_{\boldsymbol{p}\sigma} c_{\boldsymbol{p}\sigma} + \sum_{\sigma\sigma'} \int d^3 r_1 \int d^3 r_2 \; c^\dagger_{\boldsymbol{r}_1\sigma} c_{\boldsymbol{r}_1\sigma} v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) c^\dagger_{\boldsymbol{r}_2\sigma'} c_{\boldsymbol{r}_2\sigma'}, \quad (1)$$

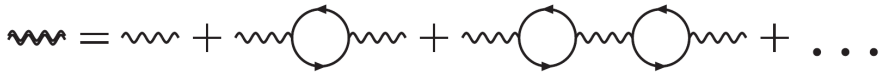
where we consider a continuous system with single-particle energy $\varepsilon_{\boldsymbol{p}} = p^2/2m$ and chemical potential $\mu = \varepsilon_F$ . The energy term is written in momentum representation, while the interaction term is written in position representation.

a) For reasons that will become obvious in problem c) below, it is convenient to write the Coulomb interaction potential in momentum representation. Show that in $d = 3$ dimensions the Fourier transform of the Coulomb potential with respect to the distance vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is $v_q = 4\pi e^2/q^2$. You can show this either by direct Fourier transformation or by observing that $v(r)$ obeys Poissons equation for a point-charge (Maxwells equation), $\nabla^2 v(r) = 4\pi e^2 \delta^3(\boldsymbol{r})$, and solving this equation in momentum representation.

b) Give the interaction term in Eq. (1) in momentum representation.

c) *Screening of the Coulomb potential.* The interaction appears as a wavy line in any diagrammatic contribution. The electron Greens functions are represented by solid lines. The renormalized interaction is denoted by $\tilde{v}_q$ and is represented by a bold wavy line. Each "electron bubble" diagram is denoted by $\Pi(q)$ and is called the polarization function.

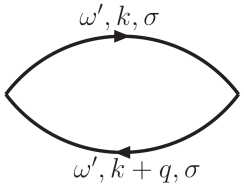
   Thus, the interaction is renormalized by the following infinite series of diagrams,



Convince yourself that this series is generated by separating any diagramatic correction to the interaction into irreducible diagrams with respect to cutting a wavy line (in analogy to the irreducible corrections of Greens functions) and then taking into account only the leading contribution in the interaction to each irreducible part. Now label each line in the above diagrams with the appropriate momentum, energy and spin variables, and evaluate the infinite series in momentum representation to obtain $\tilde{v}_q$ in terms of $\Pi(q)$. Note that this is easy in momentum representation.

### 4.2 Lindhard function (20 points)

The polarization function is represented by

$\Pi(q) =$ 


a) Show that

$$\begin{aligned}\Pi(q) &= 2\sum_{\boldsymbol{k}} \frac{f(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-\mu)-f(\varepsilon_{\boldsymbol{k}}-\mu)}{\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}-\varepsilon_{\boldsymbol{q}}} \\ &\overset{T\to 0}{=} 2\int \frac{d^dk}{(2\pi)^d} \frac{\theta\left(\mu-\varepsilon_{\boldsymbol{k}+\boldsymbol{q}/2}\right)-\theta\left(\mu-\varepsilon_{\boldsymbol{k}-\boldsymbol{q}/2}\right)}{\varepsilon_{\boldsymbol{k}+\boldsymbol{q}/2}-\varepsilon_{\boldsymbol{k}-\boldsymbol{q}/2}} \end{aligned} \tag{2}$$

b) The main contribution to the integral arises from the vanishing denominator, i.e., small momentum transfer $\boldsymbol{q}$. Therefore, neglect all terms of order $\mathrm{O}(q^2)$ in the denominator of the integrand. Show that with this approximation

$$\Pi(q) \;=\; \frac{2m}{q\pi}\int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}}\int_{k_+}^{k_-}\frac{dk_\parallel}{k_\parallel} \quad \text{with} \quad k_\pm = \sqrt{k_F^2-k_\perp^2}\pm\frac{q}{2}. \tag{3}$$

Hint: Use a coordinate system such that $\boldsymbol{k}=\left(\boldsymbol{k}_\perp, k_\parallel\right)$, where $k_\parallel$ denotes the component of $\boldsymbol{k}$ pointing in the direction of $\boldsymbol{q}$.

c) Calculate $\Pi(q)$ in $d=1,3$ dimensions and show

$$\Pi(q) \;=\; \begin{cases} \frac{m}{\pi k_F}\frac{1}{x}\ln\left|\frac{1+x}{1-x}\right|, & d=1 \\ -q_{TF}^2\frac{1}{2}\left(1+\frac{1-x^2}{2x}\ln\left|\frac{1+x}{1-x}\right|\right), & d=3 \end{cases} \tag{4}$$

where $x = q/2k_F$ and, in $d = 3$ dimensions, $q_{TF} = \sqrt{4e^2 m k_F/\pi}$ is the so-called Thomas-Fermi wave number. The function in Eq. (2) is called the Lindhard function and is shown in the figure.

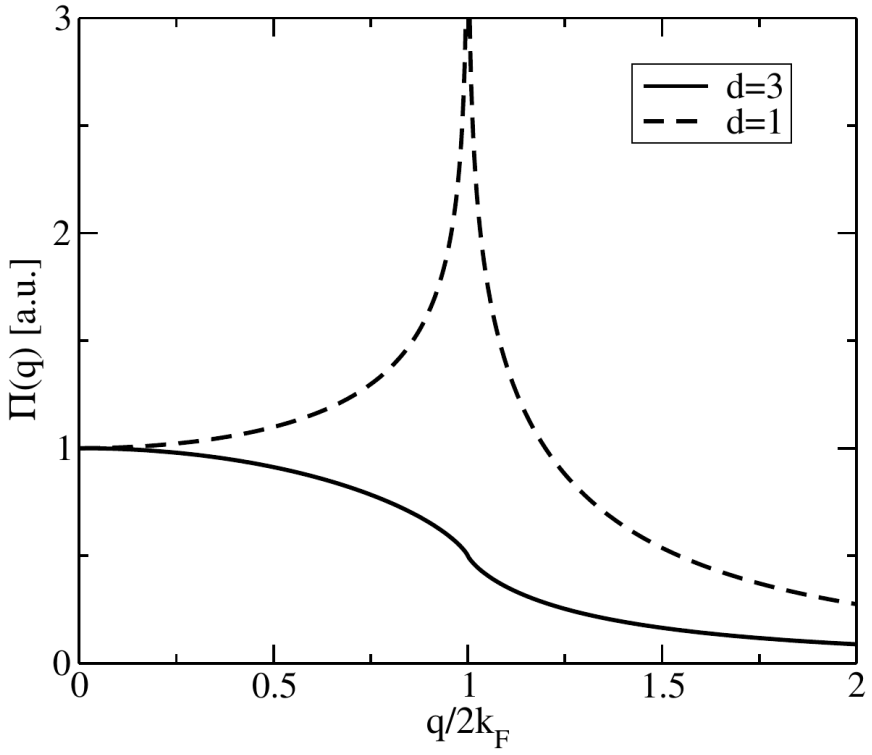



Figure 1: The Lindhard function in $d = 1, 3$ dimensions.