Universität Bonn May 2, 2018
Physikalisches Institut
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Advanced Theoretical Condensed Matter Physics
Summer term 2018

Exercise 2

(Solutions due on May 15, 2018, 12h)

2.1 Some physical properties described by single-particle Green's functions
(15 points)

The retarded/advanced single-particle Green's function of an electron system in continuous space reads in frequency and momentum representation ($\hbar = 1$),

$$G_{\mathbf{p}\sigma}^{R/A}(\omega) = \frac{1}{\omega + \mu - \varepsilon_{\mathbf{p}} - \Sigma_{\mathbf{p}\sigma}^{R/A}(\omega)} \ , \tag{1}$$

with the kinetic energy $\varepsilon_{\mathbf{p}} = p^2/2m$, the chemical potential μ (at temperature $T = 0$: $\mu = \varepsilon_F$), and the self-energy $\Sigma_{\mathbf{p}\sigma}^{R/A}(\omega)$.

We first consider the non-interacting case, i.e., $\Sigma_{\mathbf{p}}^{R/A}(\omega) = \mp \mathrm{i}\eta \to \mathrm{i}0^{\mp}$.

a) Calculate the density of states per spin orientation σ in $d = 1, 2, 3$ spatial dimensions as

$$N(\omega) = \frac{1}{\pi}\,\mathrm{Im} \int \frac{d^d p}{(2\pi)^d}\, G_{\mathbf{p}\sigma}^{A}(\omega), \tag{2}$$

including the correct normalization factor. Compare with the result of a direct enumeration of the states in a shell of constant energy in momentum space. *Result:* $N(\omega) \propto (\omega + \mu)^{(d-2)/2}$.

b) *Spatial dependence: Fourier transformation.* Show by Fourier transformation that in $d = 3$ dimensions the free Green's function in position representation has the spatial dependence

$$G_{\sigma}^{R/A}(\omega, r) = -\pi N(\mu + \omega) \frac{\mathrm{e}^{\pm ip(\mu+\omega)r}}{p(\mu + \omega)r} \ , \tag{3}$$

with the distance $r = |\mathbf{r}_2 - \mathbf{r}_1|$ and $p(\mu + \omega) = \sqrt{2m(\mu + \omega)}$ the momentum of a particle with energy $(\mu + \omega)$.

Hint: Do the $|\mathbf{p}|$-integral separately for the imaginary and real parts of $G^{R/A}$.

c) *Spatial dependence: equation of motion.* Confirm that the above result for $G_{\sigma}^{R/A}(\omega, r)$ also obeys the equation of motion of the free Green's function in ω and position representation in $d = 3$ dimensions.

We now consider a Fermi liquid of interacting electrons. In this case, the self-energy $\Sigma_{\mathbf{p}\sigma}^{R/A}(\omega)$ is a complex function of ω. However, its imaginary part vanishes at the Fermi energy, $\Sigma_{\mathbf{p}\sigma}^{R/A}(\omega \to 0) \sim \omega^2$.

d) Determine the position z_0 of the pole of $G^{R/A}(z)$ for $\omega \to 0$ as well as the residue Z of the pole by expanding $\Sigma^{R/A}_{\mathbf{p}\sigma}(\omega)$ around $\omega = 0$. What is the physical meaning of $\mathrm{Re}z_0$, $\mathrm{Im}z_0$ and Z?

e) *Momentum distribution* $n_{\mathbf{p}}$. The momentum distribution of the Fermi liquid is defined in terms of the (time-ordered) equal-time Green's function as (compare lecture),

$$n_{\mathbf{p}\sigma} = \lim_{t \to t'-0} G_{\mathbf{p}\sigma}(t, t') \tag{4}$$

Represent the equal-time Green's function as a Matsubara sum and rewrite it as a contour integral. Then show that, for $T = 0$, $n_{\mathbf{p}\sigma}$ has a discontinuous step at the Fermi momentum p_F, and that the step height is equal to the quasiparticle weight Z, as calculated above. Sketch $n_{\mathbf{p}\sigma}$ qualitatively as a function of p. The discontinuous step of $n_{\mathbf{p}\sigma}$ is a characteristic signature of a Fermi liquid.

Hint: Calculate $n_{\mathbf{p}\sigma}$ in the limits $p \to (p_F - 0)$ and $p \to (p_F + 0)$ and observe the position of the pole.

2.2 Spatial density correlations in the free Fermi gas (15 points)

Due to the Pauli principle two electrons of equal spin in a Fermi system cannot be simultaneously at the same point in space. Because the electron wave functions are continuous, this effect of Fermi statistics induces spatially extended, repulsive density correlations in a Fermi gas even without interactions, so-called *statistical correlations.* To describe this effect quantitatively, we calculate the density response function for equal spins σ,

$$\chi^R_{\sigma\sigma}(\omega, r) = -\mathrm{i}\Theta(t_1 - t_2)\langle[\rho_\sigma(t_1, \mathbf{r}_1), \rho_\sigma(t_1, \mathbf{r}_1)]\rangle|_\omega \ , \quad \rho_\sigma(t, \mathbf{r}) = \Psi^\dagger_\sigma(t, \mathbf{r})\Psi_\sigma(t, \mathbf{r}) \ , \tag{5}$$

in the static case, i.e., for $\omega = 0$, as a function of the distance $r = |\mathbf{r}_2 - \mathbf{r}_1|$ of the two particles.

a) Draw all Feynman diagrams representing $\chi^R_{\sigma\sigma}(\omega, r)$ in terms of single-particle Green's functions. Explain, in particular, why the topologically disconnected diagram does contribute in this case.

b) For each diagram, write down its mathematical expression, using the Feynman rules and the Matsubara technique. *Note:* Since we are interested in the static spatial correlations, use the representation of the single-particle Green's function in frequency (ω) and position (r) representation, as given in Eq. (2), in three dimensions. No momentum integrations are then required.

c) Evaluate the Feynman diagrams: Perform first the Matsubara summations for finite temperature T using contour integration, then continue to real frequencies.

d) Now consider the limit temperature $T = 0$. Show that the static density response function $\chi^R_{\sigma\sigma}(0, r)$ has oscillatory behavior according to

$$\chi^R_{\sigma\sigma}(\omega = 0, r) = \frac{1}{4} N(\varepsilon_F) \frac{\sin(x) - x\cos(x)}{x^4} + \mathrm{const.}, \tag{6}$$

with $x = 2p_F r$ and $r = |\mathbf{r}_2 - \mathbf{r}_1|$. Sketch $\chi^R_{\sigma\sigma}(\omega, r)$ as a function of r.

e) Which Feynman diagrams and which r-dependence would you obtain for opposite spins, $\chi^R_{\sigma,-\sigma}(0, r)$?