Universität Bonn April 18, 2018
Physikalisches Institut
Prof. Dr. Hans Kroha, Tim Lappe, Dr. Farzaneh Zamani
http://www.kroha.uni-bonn.de/teaching

Advanced Theoretical Condensed Matter Physics
Summer term 2015

Exercise 1

(Solutions due on May 2, 2018, 10h)

1.1 Green's functions for noninteracting electrons (10 points)

In the lecture the position dependent Green's function was defined in terms of field operators. In a similar way one can define the momentum dependent, retarded Green's function,

$$G^R_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\frac{1}{Z_G}\mathrm{tr}\left\{\mathrm{e}^{-\beta(H-\mu N)}\left[c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')\right]_+\right\}, \tag{1}$$

and analogously the advanced and time-ordered, momentum dependent Green's functions. We consider a system of noninteracting electrons,

$$\mathcal{H}_0 = H_0 - \mu N = \sum_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}} - \mu)c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} \tag{2}$$

a) Determine the time-dependence of $c_{\mathbf{k}\sigma}(t)$ and $c^\dagger_{\mathbf{k}\sigma}(t')$ in the Heisenberg picture.

b) Compute the retarded Green's function $G^R_{\mathbf{k}\sigma}(t,t') \equiv G^{(0)R}_{\mathbf{k}\sigma}(t-t')$ for the noninteracting system $\mathcal{H}_0$, using the results of a).

c) Calculate the Fourier transform with respect to time $(t-t')$. Discuss in particular, why an infinitesimal imaginary part $i\eta$ must be added to the energy in order to make the Fourier integral convergent and determine the sign of η.

   Result:

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d(t-t')\, G^{(0)R}_{\mathbf{k}\sigma}(t-t')\,\mathrm{e}^{\mathrm{i}\omega(t-t')} = \frac{1}{\omega+\mu-\epsilon_{\mathbf{k}}+\mathrm{i}\eta}\ , \quad \eta \to 0^+\ . \tag{3}$$

   Similarly, calculate the expression advanced Green's function $G^{(0)A}_{\mathbf{k}\sigma}(\omega)$ and determine the sign of η in this case.

For a noninteracting system, the Green's function can also be defined as the resolvent of a differential operator, in our case the Schrödinger operator,

$$(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{H}_0)G^{(0)R}(t-t') = \delta(t-t')\ . \tag{4}$$

d) Solve this equation by Fourier transforming to the frequency domain ands show that the result is identical to Eq. (3). Show, in particular, that the retarded and advanced Green's functions obey the same equation of motion. Hence, which condition determines the sign of η?

In the general, interacting case, the retarded Green's function contains a non-infinitesimal imaginary part in the denominator,

$$G^R_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega + \mu - \epsilon_{\mathbf{k}} + \mathrm{i}/(2\tau)}$$

e) Use complex contour integration to calculate the time-dependent Green's function by Fourier transformation. How does the Green's function behave for large $(t - t')$? Give a physical interpretation for τ and try to explain why a finite τ may occur.

1.2 Green's functions: general properties (10 points)

In the previous exercise, you calculated explicitely the retarded/advanced Green's function $G^{R/A}$ for the special case of a single-particle Hamiltonian. However, in the presence of interactions the system is more complex. Nevertheless some analytical properties of $G^{R/A}$ always hold. Some of them will be discussed in this exercise.

a) Normalization:
Use the definition of the spectral function $A_{\mathbf{k}\sigma}(\omega)$ (see lecture),

$$A_{\mathbf{k}\sigma}(\omega) = \frac{1}{Z_G} \sum_{n,m} |\langle n|c_{\mathbf{k}\sigma}|m\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \delta(\omega + E_n - E_m), \quad (5)$$

to show that $A_{\mathbf{k}\sigma}(\omega)$ is normalized,

$$\int_{-\infty}^{\infty} d\omega\, A_{\mathbf{k}\sigma}(\omega) = 1.$$

Use the spectral representation of $G^{R/A}_{\mathbf{k}\sigma}(\omega)$ (see lecture),

$$G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d\omega' \frac{A_{\mathbf{k}\sigma}(\omega')}{\omega - \omega' \pm \mathrm{i}0^+},$$

to find the relation between $\mathrm{Re}G^{R/A}_{\mathbf{k}\sigma}(\omega)$, $\mathrm{Im}G^{R/A}_{\mathbf{k}\sigma}(\omega)$, and $A_{\mathbf{k}\sigma}(\omega)$.
(Kramers-Kronig relation).

b) Asymptotic behavior:
You can assume that $A_{\mathbf{k}\sigma}(\omega) = 0$ for $|\omega| > \omega_{max}$ for some $\omega_{max} > 0$. (finite band width of lattice systems). Show

$$\lim_{\omega \to \pm\infty} \omega \cdot G^R_{\mathbf{k}\sigma}(\omega) = 1,$$

i.e., $G^R_{\mathbf{k}\sigma}(\omega) \to 1/\omega$ for large frequencies $|\omega| \gg 1$.