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http://www.kroha/uni-bonn.de/teaching

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Advanced Theoretical Condensed Matter Physics
Summer term 2015

Exercise 1

(Solutions due on April 30, 2015, 12h)

1.1 Green's functions for noninteracting electrons (10 points)

In the lecture the position dependent Green's function was defined. In the same way one can define a momentum dependent retarded Green's function,

$$G^R_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\frac{1}{Z_G}\mathrm{tr}\left\{\mathrm{e}^{-\beta(H-\mu N)}\left[c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')\right]_+\right\} \tag{1}$$

The advanced and time-ordered momentum dependent Green's functions are defined in analogy to the lecture. Here we will consider a system of noninteracting electrons,

$$\mathcal{H}_0 = H_0 - \mu N = \sum_{\mathbf{k}\sigma}(\epsilon_\mathbf{k} - \mu)c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}$$

a) Determine the time-dependence of $c_{\mathbf{k}\sigma}(t)$ and $c^\dagger_{\mathbf{k}\sigma}(t')$ for the noninteracting system $\mathcal{H}_0$ by using the equation of motion for a Heisenberg operator.

b) Compute the retarded Green's function (1) for the noninteracting system using the results of b).

$$G^{(0)R}_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\mathrm{e}^{-\mathrm{i}(\epsilon_\mathbf{k}-\mu)(t-t')} = G^{(0)R}_{\mathbf{k}\sigma}(t-t') \tag{2}$$

c) Derive the Fourier transform

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d(t-t')\, G^{(0)R}_{\mathbf{k}\sigma}(t-t')\,\mathrm{e}^{\mathrm{i}\omega(t-t')} = \frac{1}{\omega - (\epsilon_\mathbf{k}-\mu) + \mathrm{i}\eta}\ , \tag{3}$$

where $\eta > 0$ is an infinitesimal, positive number.

Hint: Observe and discuss that an infinitesimal imaginary part $i\eta$ must be introduced to the energy in order to make the integral convergent.

In an analogous manner, calculate the expression for the noninteracting, advanced Green's function $G^{(0)A}_{\mathbf{k}\sigma}(\omega)$.

The noninteracting Green's function can also be defined as the resolvent of a differential operator. In our case this is the Schrödinger operator,

$$(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{H}_0)G^{(0)R}(t-t') = \delta(t-t')$$

d) Show that this equation is fullfilled by Eq. (2) by writing it in momentum space representation. What is the corresponding equation in energy space? Show that (3) is the resolvent of the Schröedinger operator in energy space.

In the general, interacting case, the retarded Green's function contains a non-infinitesimal imaginary part in the denominator,

$$G^R_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + \mathrm{i}/(2\tau)}$$

e) Use the residue theorem to calculate the time-dependent Green's function by Fourier transform. How does the Green's function behave for large $(t-t')$? Give a physical interpretation for τ and try to explain why a finite τ may occur.

1.2 Green's functions: general properties (10 points)

In the previous exercise, you calculated explicitely the retarded Green's function G for the special case of a diagonal Hamiltonian. However, in most cases the system is more complex, e.g., in the presence of interactions. Nevertheless some analytical properties of G will always hold. Some of them will be discussed in this exercise.

a) Normalization:
Use the explicit definition of the spectral function $A_{\mathbf{k}\sigma}(\omega)$ (see lecture),

$$A_{\mathbf{k}\sigma}(\omega) = \frac{1}{Z_G} \sum_{n,m} |\langle n|c_{\mathbf{k}\sigma}|m\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \delta(\omega + E_n - E_m), \qquad (4)$$

to show that $A_{\mathbf{k}\sigma}(\omega)$ is normalized,

$$\int_{-\infty}^{\infty} d\omega\, A_{\mathbf{k}\sigma}(\omega) = 1.$$

Use the spectral representation of $G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega)$ (see lecture),

$$G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d\omega' \frac{A_{\mathbf{k}\sigma}(\omega')}{\omega - \omega' + \mathrm{i}0^+},$$

to find the relation between $\mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega)$ and $A_{\mathbf{k}\sigma}(\omega)$. Calculate

$$\int_{-\infty}^{\infty} d\omega\, \mathrm{Im}G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega).$$

b) Asymptotic behavior:
You can assume that $A_{\mathbf{k}\sigma}(\omega) = 0$ if $|\omega| > \omega_{max}$ for some $0 < \omega_{max} < \infty$. (finite band width of lattice systems). Show

$$\lim_{\omega\to\pm\infty} \omega \cdot G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) = 1,$$

i.e., $G^{\mathrm{R}}_{\mathbf{k}\sigma}(\omega) \approx 1/\omega$ for large energies.