1.3 The general form of the Green's function in frequency space

1.3.1 The free Green's function (fermions)

We consider a system of free fermions with the Hamiltonian

$$H = \sum_{p\sigma} \varepsilon_p \, c^+_{p\sigma} c_{p\sigma}$$

From the equation of motion of the operators,

$$i \frac{d}{dt} c_{p\sigma} = \left[c_{p\sigma}, H - \mu N \right] = (\varepsilon_p - \mu)\, c_{p\sigma} \, ,$$

one obtains the equation of motion (EOM) of the Green's function,

$$G_{p\sigma}(t) = -i \left\langle\!\left\langle T \left\{ c_{p\sigma}(t)\, c^+_{p\sigma}(0) \right\} \right\rangle\!\right\rangle$$

$$\text{with } \langle\!\langle \ldots \rangle\!\rangle = tr \left\{ \frac{e^{-\beta(H-\mu N)}}{Z_G} \ldots \right\}$$

(which is diagonal in $\vec{p}, \sigma$ for a translation and spin rotation invariant system)

as

$$\boxed{\left(i \frac{d}{dt} + \mu - \varepsilon_p \right) G_{p\sigma}(t) = \delta(t) .}$$

The "$\delta(t)$" arises from the differentiation of the $\theta(\pm t)$ functions involved in the time ordering.

Note that the EOM is the same for G, G^R, G^R.
Only the different boundary conditions in time distinguish the three types of solutions.

By Fourier transformation one obtains at $T=0$

$$G_{p\sigma}(\omega) = \frac{1}{\omega+\mu-\varepsilon_p + i\eta\, \mathrm{sgn}(\varepsilon_p-\mu)}$$

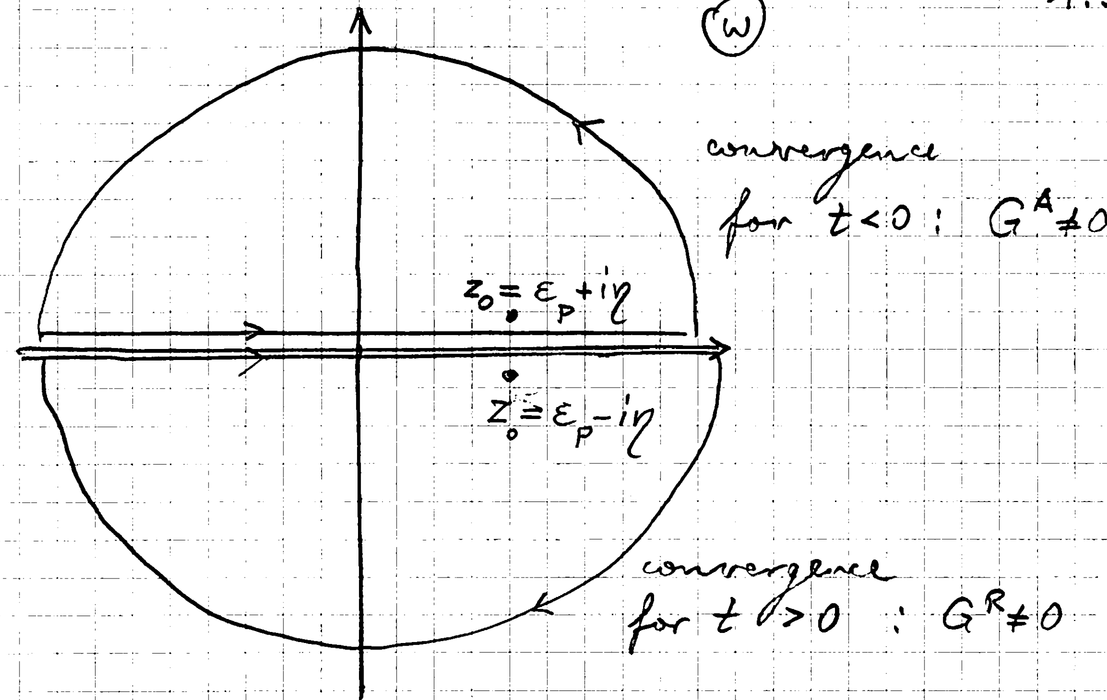
This is in agreement with the spectral representation, where for the non-interacting system the spectral function is $A(\vec{p},\omega) = \delta(\omega+\mu-\varepsilon_p)$.

The free retarded G-functions are (for any T)

$$G^{R/A}_{p\sigma}(\omega) = \frac{1}{\omega+\mu-\varepsilon_p \pm i\eta},$$

again in agreement with the spectral representation.

The fact that $G^{R/A}(t) \neq 0$ for $t \gtrless 0$ requires via contour integration that in the ω-domain $G^R(\omega)$ / $G^A(\omega)$ has a pole in the lower/upper ω half plane, respectively.
This is obeyed by the above expression for $G^{R/A}$ and is in accordance with the analyticity of $G^{R/A}(\omega)$ in the upper / lower half plane.



$$G^{R/A}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, G^{R/A}(\omega)$$

1.3.2 The physical meaning of the pole

We will see by explicit calculation ($\rightarrow$ perturbation theory) that due to interactions the pole of $G^{R/A}(\omega)$ at $\omega = z_0$ is shifted away from the real axis, i.e. z_0 acquires a finite imaginary part, given by the selfenergy $\Sigma^{R/A}_{p\sigma}(\omega)$ which is in general a function of $\vec{p}$ and ω and in a magnetic system also of σ.
In an interacting system the R/A Green's function hence takes the form

$$\boxed{G^{R/A}_{\vec{p}\sigma}(\omega) = \frac{1}{\omega + \mu - \varepsilon_p - \Sigma^{R/A}_{p\sigma}(\omega)}}$$

with $\mathrm{Im}\,\Sigma^{R/A} \lessgtr 0$