In this exercise sheet we want to ~~prove~~ prove the relations: ①

① $G^R = G^{--} - G^{-+} = G^{+-} - G^{++}$

$G^A = G^{--} - G^{+-} = G^{-+} - G^{++}$

$G = G^{--}$

② $\Sigma^{++} + \Sigma^{--} = -(\Sigma^{+-} + \Sigma^{-+}) =: \Omega$

①: We recall the definitions: $iG^{--}_{12} = \langle \hat{T}\hat{\Psi}_1 \hat{\Psi}^+_2 \rangle$, $iG^{+-}_{12} = \langle \hat{\Psi}_1 \hat{\Psi}^+_2 \rangle$

$iG^{++}_{12} = \langle \hat{\tilde{T}} \hat{\Psi}_1 \hat{\Psi}^+_2 \rangle$, $iG^{-+}_{12} = \mp \langle \hat{\Psi}^+_2 \hat{\Psi}_1 \rangle$

T: time-ordering operator, $\hat{\tilde{T}}$ = inverse time-ordering operator

In addition, we have the fact that $G^{--} + G^{++} = G^{-+} + G^{+-}$.

and the definition of the retarded and advanced G.F:

$$iG^R_{12} = \begin{cases} \langle \hat{\Psi}_1 \hat{\Psi}^+_2 \pm \hat{\Psi}^+_2 \hat{\Psi}_1 \rangle, & t_1 > t_2 \\ 0, & t_1 < t_2 \end{cases}$$

$$iG^A_{12} = \begin{cases} 0, & t_1 > t_2 \\ -\langle \hat{\psi}_1 \hat{\psi}_2^+ \pm \hat{\psi}_2^+ \hat{\psi}_1 \rangle, & t_1 < t_2 \end{cases}$$

Hence: $G^{--} - G^{-+} = \langle \hat{T} \hat{\psi}_1 \hat{\psi}_2^+ \rangle \pm \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle = \left[\theta(t_1 - t_2) \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle \mp \theta(t_2 - t_1) \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle \right]$

$$\pm \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle$$

$t_1 > t_2 \curvearrowright \; = \langle \hat{\psi}_1 \hat{\psi}_2^+ \pm \hat{\psi}_2^+ \hat{\psi}_- \rangle$ (since 2^{nd} term in square brackets $= 0$)

$t_1 < t_2 \nearrow \; = \mp \langle \hat{\psi}_2^+ \psi_1 \rangle \pm \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle = 0$

$$\equiv iG^R_{12}$$

$G^{+-} - G^{++} = \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle - \langle \tilde{\hat{T}} \hat{\psi}_1 \hat{\psi}_2^+ \rangle = \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle - \left[\theta(t_2 - t_1) \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle \right.$

$$\left. \mp \theta(t_1 - t_2) \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle \right]$$

$t_1 > t_2 \curvearrowright \; = \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle \pm \langle \hat{\psi}_2^+ \hat{\psi}_1 \rangle$

$t_1 < t_2 \curvearrowright \; = \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle - \langle \hat{\psi}_1 \hat{\psi}_2^+ \rangle = 0.$

and similarly for the iG^A_{12}

② $\Sigma^{++} + \Sigma^{--} = -(\Sigma^{+-} + \Sigma^{-+})$

We start from the Dyson's equation $G_{12} = G^{(0)}_{12} + \int d^4x_3\, d^4x_4\, G^{(0)}_{14} \Sigma_{43} G_{32}$ ⊛

where $G_{12} = \begin{pmatrix} G^{--}_{12} & G^{-+}_{12} \\ G^{+-}_{12} & G^{++}_{12} \end{pmatrix}$, $\Sigma_{43} = \begin{pmatrix} \Sigma^{--}_{43} & \Sigma^{-+}_{43} \\ \Sigma^{+-}_{43} & \Sigma^{++}_{43} \end{pmatrix}$

we know the general identity: $G^{--} + G^{++} = G^{-+} + G^{+-}$

$\Rightarrow G^{--} + G^{++} - G^{-+} - G^{+-} = 0. \Rightarrow \hat{G}^{-1}_{01}(G^{--} + G^{++} - G^{-+} - G^{+-}) = 0.$

We apply the operator $\hat{G}^{-1}_{01}$ to ⊛: $\hat{G}^{-1}_{01} \hat{G}_{12} = \hat{G}^{-1}_{01} \hat{G}^{(0)} + \int d^4x_3\, d^4x_4\, \hat{G}^{-1}_{01} G^{(0)}_{14} \Sigma_{43} G_{32}$

note: $\hat{G}^{-1}_{01}$ is the inverse of the free Green's fctn operator, and $\hat{G}^{-1}_{01}$ means to act on the variable labeled "1" only. $(\vec{r}_1, t_1)$. $\hat{G}^{-1}_{01}$ obeys

$$\hat{G}^{-1}_{01} \hat{G}^{(0)}_{12} = \hat{\sigma}_z \delta(X_1 - X_2) = \delta_{\sigma_1 \sigma_2} \delta(t_1 - t_2) \delta(\vec{r}_1 - \vec{r}_2)$$

Hence $\hat{G}^{-1}_{01} G_{12} = \hat{\sigma}_z \delta(x_1 - x_2) + \int d^4x_3\, d^4x_4\, \delta(X_1 - X_4) \Sigma_{43} G_{32}$

rules of $\hat{G}^{-1}_{01}$: $\hat{G}^{-1}_{01} G^{(0)--}_{12} = \delta(x_1 - x_2)$, $\hat{G}^{-1}_{01} G^{(0)++}_{12} = -\delta(x_1 - x_2)$

$\hat{G}^{-1}_{01} G^{(0)+-}_{12} = \hat{G}^{-1}_{01} G^{(0)-+}_{12} = 0.$

Hence
$$\hat{G}_{01}^{-1}\begin{pmatrix} G^{--}_{12} & G^{-+}_{12} \\ G^{+-}_{12} & G^{++}_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(x_1-x_2) + \int d^4x_3\, d^4x_4 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(x_1-x_4)$$
$$\times \underbrace{\begin{pmatrix} \Sigma^{--}_{43} & \Sigma^{-+}_{43} \\ \Sigma^{+-}_{43} & \Sigma^{++}_{43} \end{pmatrix}\begin{pmatrix} G^{--}_{32} & G^{-+}_{32} \\ G^{+-}_{32} & G^{++}_{32} \end{pmatrix}}$$

$$\begin{pmatrix} \Sigma^{--}_{43} & \Sigma^{-+}_{43} \\ \Sigma^{+-}_{43} & \Sigma^{++}_{43} \end{pmatrix}\begin{pmatrix} G^{--}_{32} & G^{-+}_{32} \\ G^{+-}_{32} & G^{++}_{32} \end{pmatrix} = \begin{pmatrix} \Sigma^{--}_{43}G^{--}_{32} + \Sigma^{-+}_{43}G^{+-}_{32} & \Sigma^{--}_{43}G^{-+}_{32} + \Sigma^{-+}_{43}G^{++}_{32} \\ \Sigma^{+-}_{43}G^{--}_{32} + \Sigma^{++}_{43}G^{+-}_{32} & \Sigma^{+-}_{43}G^{-+}_{32} + \Sigma^{++}_{43}G^{++}_{32} \end{pmatrix}$$

Now
$$\hat{G}_{01}^{-1}\left(G^{--} + G^{++} - G^{-+} - G^{+-}\right) = 0 = \hat{G}_{01}^{-1}\Big(\Sigma^{--}_{13}G^{--}_{32} + \Sigma^{-+}_{13}G^{+-}_{32} - \Sigma^{+-}_{13}G^{-+}_{32} - \Sigma^{++}_{13}G^{++}_{32}$$
$$- \Sigma^{--}_{13}G^{-+}_{32} - \Sigma^{-+}_{13}G^{++}_{32} + \Sigma^{+-}_{13}G^{--}_{32} + \Sigma^{++}_{13}G^{+-}_{32}\Big)$$

$$\Rightarrow \Sigma^{++}_{13}\left(-G^{++}_{32} + G^{+-}_{32}\right) + \Sigma^{--}_{13}\left(G^{--}_{32} - G^{-+}_{32}\right) + \Sigma^{-+}_{13}\left(G^{+-}_{32} - G^{++}_{32}\right) + \Sigma^{+-}_{13}\left(-G^{-+}_{32} + G^{--}_{32}\right) = 0$$

$$\Rightarrow \left(\Sigma^{++}_{13} + \Sigma^{-+}_{13}\right)\left(-G^{++}_{32} + G^{+-}_{32}\right) + \left(\Sigma^{--}_{13} + \Sigma^{+-}_{13}\right)\left(G^{--}_{32} - G^{-+}_{32}\right) = 0$$

and since obviously $G^{+-} - G^{++} = G^{--} - G^{-+}$, hence

$$\left(\Sigma^{++} + \Sigma^{-+}\right) + \left(\Sigma^{--} + \Sigma^{+-}\right) = 0 \Rightarrow \Sigma^{++} + \Sigma^{--} = -\left(\Sigma^{-+} + \Sigma^{+-}\right)$$