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Equilibrium and Non-equilibrium QFT in Many-Body Systems SS 2013

Exercise Sheet 4

(To be handed in on the 25th of June and discussed on the 27th and 28th of June)

4.1 Electron-electron interactions

Consider a system of electrons which interact via the Coulomb interaction $V(\mathbf{r}-\mathbf{r}') = \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}$:

$$H = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\sigma\sigma'} \int d^3r \int d^3r'\, \psi^\dagger_\sigma(\mathbf{r}) \psi^\dagger_{\sigma'}(\mathbf{r}')\, \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\, \psi_{\sigma'}(\mathbf{r}')\psi_\sigma(\mathbf{r}) \ , \tag{1}$$

where $c^\dagger_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}$ and $\psi^\dagger_\sigma(\mathbf{r}), \psi_\sigma(\mathbf{r})$ are creation and destruction operators for an electron in momentum eigenstate $|\mathbf{k}\sigma\rangle$ and position eigenstate $|\mathbf{r}\sigma\rangle$, respectively.

a) Show that the Fourier transform of $V(\mathbf{r}-\mathbf{r}')$ w.r.t. $(\mathbf{r}-\mathbf{r}')$ is

$$V_{\mathbf{q}} = \frac{4\pi e^2}{q^2} \ .$$

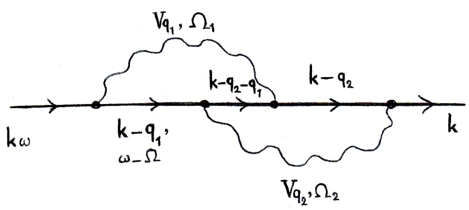
(Note: This is most conveniently shown by solving the Poisson equation for the Coulomb potential in momentum space.)

b) Derive the Hamiltonian H with the interaction term in momentum representation, *i.e.* in terms of $c^\dagger_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}$:

$$H = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\sigma\sigma'} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} c^\dagger_{\mathbf{k}-\mathbf{q},\sigma} c^\dagger_{\mathbf{k}'+\mathbf{q},\sigma'}\, V_{\mathbf{q}}\, c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma} \tag{2}$$

c) Draw all Feynman diagrams for the <u>self-energy</u> up to the 3rd order in the interaction $V_{\mathbf{q}}$ and label all lines with the appropriate momentum and energy labels.

d) Write down the mathematical expression for the following Feynman diagram; there is no need to perform the Matsubara summations.



4.2 Phonons II: Electron-Phonon Interaction and Self-Energy

We continue our discussion of phonons from the previous exercise sheet. We now consider electron-phonon interactions, which can be described by a term in the Hamiltonian describing coupling of electrons to a positive ionic background

$$H_{e-p} = \int d\mathbf{r}\,(-e)\rho_{el}(\mathbf{r}) \sum_{j=1}^{N} V_{ion}(\mathbf{r}-\mathbf{r}_j)\,, \tag{3}$$

where $\rho_{el}(\mathbf{r})$ is the electronic density at $\mathbf{r}$, $\mathbf{r}_i = \mathbf{R}_i^0 + \mathbf{u}_j$ where $\mathbf{u}_j$ is the displacement of the atom at site j away from the equilibrium position $\mathbf{R}_j^0$, and $V_{ion}(\mathbf{r}-\mathbf{r}_j)$ the ionic potential.

a) We want to derive the second-quantized form of H_{e-p} in momentum representation. To do this first Taylor-expand $V_{ion}(\mathbf{r}-\mathbf{r}_j)$ to first order in $\mathbf{u}_j$, and consider only the term proportional to u_j, where we now take an isotropic media and take the displacement u_j in only one direction. Use $V_{ion}(\mathbf{r}) = \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} V_{\mathbf{p}} e^{i\mathbf{k}\cdot\mathbf{r}}$ and $u_j = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_j^0}$, $u_{\mathbf{k}}$ have been defined in the previous sheet.
Hint: For normalization purposes, use $\sum_j e^{i\mathbf{k}\cdot\mathbf{R}_j^0} = N\delta_{\mathbf{k},0}$ *and* $\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} = V\delta_{\mathbf{k},0}$

b) From the lecture, we know that the Matsubara Green's function can be written in the compact form

$$\mathcal{G}_{\mathbf{p}}(\tau) = \sum_{n=0}^{\infty}\int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n \langle T_\tau H_{int}(\tau_1)H_{int}(\tau_2)\ldots H_{int}(\tau_n) c_{\mathbf{p}\sigma}(\tau) c^\dagger_{\mathbf{p}\sigma}(0)\rangle_{0,con-dif} \tag{4}$$

where the average $\langle\ldots\rangle_{0,con-dif}$ is to be done with respect to the noninteracting Hamiltonian

$$H_0 = H_{el} + H_{ph} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} \left(b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + \frac{1}{2}\right) \tag{5}$$

and includes only the *different*, *connected* diagrams. Use the form of H_{int} derived in problem a) and applying Wick's theorem, write down all possible terms for order $n = 2$.
Hint: There are 6 *terms in total. Why do some of these terms vanish?*

c) Evaluate the following self-energy diagram in the Matsubara formalism. The straight and wavy lines denote the electron and phonon propagators, respectively. Note that the values denoted on the diagram should be interpreted as 4-vectors, i.e., $k = (\mathbf{k}, ik_n) \equiv$ (momentum, Matsubara frequency).