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Equilibrium and Non-equilibrium QFT in Many-Body Systems SS 2013

Exercise Sheet 2

(To be handed in on 21. May and discussed on 23. and 24. May)

2.1 Contour integration: local Green's function on a lattice

The free, local (i.e. at position $\mathbf{r} = 0$) retarded Green's function is defined as

$$G_0^R(\omega) = \sum_{\mathbf{k}} \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\eta}\,. \tag{1}$$

We want to calculate $G_0^R(\omega)$ for a lattice in $d = 3$ dimensions with a density of states

$$N(E) = \frac{1}{\pi}\sqrt{1 - (E/D)^2}\,, \tag{2}$$

where D is the half band width. In the following, we use units $D = 1$.

a) Show that the expression (1) can be rewritten as,

$$G_0^R(\omega) = \int_{-1}^{+1} dE\ \frac{N(E)}{\omega - E + i\eta}\,. \tag{3}$$

b) Transform Eq. (3) into a contour integral around the complex unit circle ($|z| = 1$) by substituting first $E = \cos\varphi$ and then $z = e^{i\varphi}$.

c) Find all poles of the resulting expression which fall inside the unit circle (observe $\eta > 0$) and use Cauchy's theorem to show that

$$G_0^R(\omega) = \omega - \sqrt{\omega^2 - 1}\,, \qquad -\infty < \omega < \infty\,. \tag{4}$$

d) Sketch real and imaginary parts of $G_0^R(\omega)$. What is, in particular, the behavior for $\omega \to 1 \pm 0$ and for $\omega \to \pm\infty$?

2.2 Normalization of the spectral Function

Show that the spectral function, $A_{\mathbf{p}}(\omega)$, is normalized (i.e. the probability for having a state $|\mathbf{p}\rangle$ at any energy ω is one),

$$\int_{-\infty}^{+\infty} d\omega\, A_{\mathbf{p}}(\omega) = 1\,. \tag{5}$$

Hint: Observe that Eq. (5) is just a Fourier transform with respect to ω for time $t = 0$. Then use the definition of the spectral function from the lecture and the equal-time commutation relations of the creation and destruction operators for both cases, fermions and bosons.

2.3 Atomic limit of the Hubbard Model

We consider the *fermionic* Hubbard model ($U > 0$, chemical potential μ),

$$H = \sum_{\substack{<ij> \\ \sigma}} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{i} \widehat{n}_{i\uparrow}\, \widehat{n}_{i-\downarrow}\,, \tag{6}$$

in the limit where the hopping matrix element t_{ij} vanishes, i.e., where all sites are decoupled from each other ("atomic limit") and only the local interaction term remains. We want to calculate the (retarded) Green's function on site j, $G^R_{j\sigma}(\omega)$, for this case. Convince yourself that, for $t_{ij} = 0$, $G_{j\sigma}(\omega)$ is diagonal in position and in spin space.

a) Starting from the general definition of the Green's function

$$G^R_{j\sigma}(t,t') = -i\Theta(t)\langle\{c_{j\sigma}(t), c^{\dagger}_{j\sigma}(t')\}\rangle\,, \tag{7}$$

where $\langle \dots \rangle$ denotes the thermal average and quantum mechanical expectation value, derive the equation of motion for $G^R_{j\sigma}(t,t')$ w.r.t. t.

b) This equation of motion contains an average of the form, $\langle\{\widehat{n}_{j-\sigma} c_{j\sigma}(t),\ c^{\dagger}_{j\sigma}(t')\}\rangle$. Use the fact that *in the atomic limit* the on-site particle number $\hat{n}_{j-\sigma}$ is a conserved quantity (i.e., $\widehat{n}_{j-\sigma}$ may be replaced by its eigenvalue when acting on an eigenstate of H) to evaluate this avarage. Then show by Fourier transforming the equation of motion to the frequency domain that

$$G^R_{j\sigma}(\omega) = \frac{1 - \langle\widehat{n}_{j-\sigma}\rangle}{\omega + \mu + i\eta} + \frac{\langle\widehat{n}_{j-\sigma}\rangle}{\omega + \mu - U + i\eta}\,. \tag{8}$$

c) Interpret this result physically. Sketch the density of states of the fermionic Hubbard model in the atomic limit.