Universität Bonn 26.04.2013
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Equilibrium and Non-equilibrium QFT in Many-Body Systems SS 2013

Exercise Sheet 1

(To be handed in on 7. May and discussed on 9. and 10. May)

1.1 The Residual Theorem

Show the following with the help of the residue theorem:

a)

$$\oint_{|z|=3} \frac{1}{z^4 + z^3 - 2z^2} dz = 0$$

b)

$$\int_0^\infty \frac{1}{(x^2+1)^2} dx = \frac{\pi}{4}$$

1.2 Green's Function in Real Space

We consider the free retarded Green's function in momentum space

$$G_0^R(\mathbf{k}, \omega) = \frac{1}{\omega + \mu - \varepsilon_{\mathbf{k}} + i\eta} \ , \tag{1}$$

where ω is the frequency and η an infinitesimal real parameter. In this problem we consider a quadratic dispersion, i.e., $\varepsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$, and we set $\mu = \varepsilon_F$, the Fermi energy.

We want to calculate the free Green's function in real space by performing a Fourier transformation on $G_0^R(\mathbf{k}, \omega)$, i.e., by evaluating the following expression:

$$G_0^R(\mathbf{r}, \omega) = \int d^3\mathbf{k} \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega + \mu - \varepsilon_{\mathbf{k}} + i\eta} \ , \tag{2}$$

a) Show, via a suitable change of integral coordinates and the introduction of dimensionless variables ν, κ and ρ, that one can write the integral in (2) in the form

$$G_0^R(\rho, \Omega) = \frac{4\pi m k_F}{i\rho} \int_{-\infty}^{\infty} d\kappa \ \kappa \, \frac{e^{i\kappa\rho}}{\Omega - \kappa^2 + i\eta} \ , \tag{3}$$

where $\Omega = \nu + 1$, $\nu = \omega/\varepsilon_F$, $\kappa = k/k_F$, and $\rho = k_F r$.

b) We distinguish between the cases $\Omega > 0$ and $\Omega < 0$. Choosing the contour such that the integral converges, calculate (3) using the residue theorem.

1.3 Interactions and Quasiparticles

Due to interactions in a many-body system, the Green's function acquires a self-energy $\Sigma(\omega) = \Sigma'(\omega) + i\Sigma''(\omega)$.

We consider an interacting Green's function of the form

$$G(\omega) = \frac{1}{\omega + \mu - \varepsilon_{\mathbf{k}} - \Sigma(\omega)} \, ,$$

where we have dropped momentum and spin dependencies in order to simplify the problem.

a) Assuming that $\Sigma(\omega)$ is an analytic function, Taylor-expand $\Sigma(\omega)$ around ω_0 up to the first order in ω, where ω_0 is a pole of $G(\omega)$, i.e.,

$$\omega_0 + \mu - \varepsilon_{\mathbf{k}} - \Sigma'(\omega_0) = 0$$

Show that the interacting Green's function $G(\omega)$ can be approximated as

$$G(\omega) \approx \frac{Z(\omega_0)}{\omega + \mu - \widetilde{\varepsilon}_{\mathbf{k}}(\omega_0) + \frac{i}{\tau(\omega)}} \, , \tag{4}$$

where $Z^{-1}(\omega_0) = 1 - \omega\partial_\omega\Sigma'(\omega)|_{\omega_0}$ is called the "quasi-particle weight". Write down the explicit forms for $\widetilde{\varepsilon}_{\mathbf{k}}(\omega_0)$ and $\tau(\omega)$.

b) Sketch the real and imaginary parts of (4).

c) Interpreting the Green's function as a transition amplitude, provide a physical interpretation for $\widetilde{\varepsilon}_{\mathbf{k}}(\omega_0)$, $\tau(\omega_0)$ and $Z(\omega_0)$.