Supplementary Material for "Inflationary quasiparticle creation and thermalization dynamis in coupled Bose-Einstein condensates"

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This supplementary material covers

1. Quantum kinetic equations in the presence of inelastic collisions
2. Evaluation of the collisional second order self-energies and derivation of equations of motion for the spectral and statistical functions
3. Notes on numerical calculations
4. Expressions for the total energy of the subsystem of Bose-Einstein condensates.

I. KINETIC EQUATIONS IN THE PRESENCE OF INELASTIC COLLISIONS IN THE TRAP EIGENBASIS. FIRST ORDER SELF-ENERGIES

From the Hamiltonian we can now derive kinetic equations of motion for the Green's functions of the condensate $\mathbf{C}(t,t')$ and quasi-particles $\mathbf{G}(t,t')$ in the trap eigenbasis. The Green's functions are defined as follows

$$\mathbf{C}_{\alpha\beta}(t,t') = -i\begin{pmatrix} a_\alpha(t)a^*_\beta(t') & a_\alpha(t)a_\beta(t') \\ a^*_\alpha(t)a^*_\beta(t') & a^*_\alpha(t)a_\beta(t')\end{pmatrix}, \quad \text{(S1)}$$

$$\begin{aligned}\mathbf{G}_{nm}(t,t') &= -i\begin{pmatrix} \langle T_C\hat{b}_n(t)\hat{b}^\dagger_m(t')\rangle & \langle T_C\hat{b}_n(t)\hat{b}_m(t')\rangle \\ \langle T_C\hat{b}^\dagger_n(t)\hat{b}^\dagger_m(t')\rangle & \langle T_C\hat{b}^\dagger_n(t)\hat{b}_m(t')\rangle\end{pmatrix} \\ &= \begin{pmatrix} G_{nm}(t,t') & F_{nm}(t,t') \\ \overline{F}_{nm}(t,t') & \overline{G}_{nm}(t,t')\end{pmatrix}. \end{aligned} \quad \text{(S2)}$$

The Dyson equations for these Green's functions have a standard form [2]. For the condensate Green's function it reads

$$\int_{-\infty}^{\infty} d\bar{t}\,\left[\mathbf{G}^{-1}_{0,\alpha\gamma}(t,\bar{t}) - \mathbf{S}^{HF}_{\alpha\gamma}(t,\bar{t})\right]\mathbf{C}_{\gamma\beta}(\bar{t},t') = \quad \text{(S3)}$$

$$-i\int_{-\infty}^{t} d\bar{t}\boldsymbol{\gamma}_{\alpha\gamma}(t,\bar{t})\mathbf{C}_{\gamma\beta}(\bar{t},t').$$

Here for convienice we write the self-energy as a sum of two terms: $\mathbf{S}^{cond}_{\alpha\beta} = \mathbf{S}^{HF}_{\alpha\beta} + \mathbf{S}_{\alpha\beta}$, where $\mathbf{S}^{HF}_{\alpha\beta}$ are the first order contributions (Bogoliubov-Hartree-Fock), whereas $\boldsymbol{\gamma}_{\alpha\beta} = \boldsymbol{S}^>_{\alpha\beta} - \boldsymbol{S}^<_{\alpha\beta}$ are contributions responsible for collisions and therefore for equilibration of the system. All senf-energies, including later appearing $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ have the same 2×2 structure in the Bogoliubov space

$$\mathbf{S}(t,t') = \begin{pmatrix} S^G(t,t') & S^F(t,t') \\ S^{\overline{F}}(t,t') & S^{\overline{G}}(t,t')\end{pmatrix}. \quad \text{(S4)}$$

The bare propagator is given by

$$\mathbf{G}^{-1}_{0,\alpha\beta}(t,t') = \left[i\tau_3\delta_{\alpha\beta}\frac{\partial}{\partial t} - \mathbb{1}E_{\alpha\beta}\right]\delta(t-t')\ , \quad \text{(S5)}$$

with $E_{11} = E_{22} = E_0$ and $E_{12} = E_{21} = -J$.

Instead of dealing with the quasi-particle Green's function $\mathbf{G}$, it is more conviniet to work with spectral function $\mathbf{A}$ and statistical function $\mathbf{F}$, which can be readily expressed via (S2) as $\mathbf{A}_{nm} = i(\mathbf{G}^>_{nm} - \mathbf{G}^<_{nm})$ and $\mathbf{F}_{nm} = (\mathbf{G}^>_{nm} + \mathbf{G}^<_{nm})/2 = \mathbf{G}^K_{nm}/2$ (with $\mathbf{G}^K_{nm}$ being the Keldysh component of (S2), and $\mathbf{G}^<_{nm}$ and $\mathbf{G}^>_{nm}$ being the so-called lesser and greater components of the Green's function (S2) [1]):

$$\mathbf{A}_{nm}(t,t') = \begin{pmatrix} \mathrm{A}^G_{nm}(t,t') & \mathrm{A}^F_{nm}(t,t') \\ \mathrm{A}^{\overline{F}}_{nm}(t,t') & \mathrm{A}^{\overline{G}}_{nm}(t,t')\end{pmatrix}, \quad \text{(S6)}$$

$$\mathbf{F}_{nm}(t,t') = \begin{pmatrix} \mathrm{F}^G_{nm}(t,t') & \mathrm{F}^F_{nm}(t,t') \\ \mathrm{F}^{\overline{F}}_{nm}(t,t') & \mathrm{F}^{\overline{G}}_{nm}(t,t')\end{pmatrix}. \quad \text{(S7)}$$

Hereafter Greek indeces, α, $\beta = 1, 2$, refer to the condensates in the left and right wells, and latin indeces, $n, m = 1, 2, \ldots, M$, denote the QP levels, we also imply Einstein summation.

It is important to note that the functions (S11) and (S7) obey useful symmetry relations

$$\begin{aligned}\mathrm{A}^{\overline{G}}(t,t') &= -\mathrm{A}^G(t,t')^* = -\mathrm{A}^G(t',t), \\ \mathrm{A}^{\overline{F}}(t,t') &= -\mathrm{A}^F(t,t')^* = \mathrm{A}^F(t',t)^*, \\ \mathrm{F}^{\overline{G}}(t,t') &= -\mathrm{F}^G(t,t')^* = \mathrm{F}^G(t',t), \\ \mathrm{F}^{\overline{F}}(t,t') &= -\mathrm{F}^F(t,t')^* = -\mathrm{F}^F(t',t)^*. \end{aligned} \quad \text{(S8)}$$

The Dyson equations for the spectral and statistical functions are

$$\int_{-\infty}^{\infty} d\bar{t}\left[\mathbf{G}^{-1}_{0,n\ell}(t,\bar{t})-\mathbf{\Sigma}^{HF}_{n\ell}(t,\bar{t})\right]\mathbf{A}_{\ell m}(\bar{t},t') = -i\int_{t'}^{t} d\bar{t}\,\mathbf{\Gamma}_{n\ell}(t,\bar{t})\mathbf{A}_{\ell m}(\bar{t},t') \quad \text{(S9)}$$

$$\int_{-\infty}^{\infty} d\bar{t}\left[\mathbf{G}^{-1}_{0,n\ell}(t,\bar{t})-\mathbf{\Sigma}^{HF}_{n\ell}(t,\bar{t})\right]\mathbf{F}_{\ell m}(\bar{t},t') = -i\Big[\int_{-\infty}^{t} d\bar{t}\,\mathbf{\Gamma}_{n\ell}(t,\bar{t})\mathbf{F}_{\ell m}(\bar{t},t') - \int_{-\infty}^{t'} d\bar{t}\,\mathbf{\Pi}_{n\ell}(t,\bar{t})\mathbf{A}_{\ell m}(\bar{t},t')\Big], \quad \text{(S10)}$$

where $\mathbf{\Pi}_{nm}=(\mathbf{\Sigma}^{>}_{nm}+\mathbf{\Sigma}^{<}_{nm})/2$ and $\mathbf{\Gamma}_{nm}=\mathbf{\Sigma}^{>}_{nm}-\mathbf{\Sigma}^{<}_{nm}$ and are 2×2 matrices in the Bogoluibov space

$$\mathbf{\Gamma}_{nm}(t,t') = \begin{pmatrix}\Gamma^{G}_{nm}(t,t') & \Gamma^{F}_{nm}(t,t')\\ \Gamma^{\overline{F}}_{nm}(t,t') & \Gamma^{\overline{G}}_{nm}(t,t')\end{pmatrix}, \quad \text{(S11)}$$

$$\mathbf{\Pi}_{nm}(t,t') = \begin{pmatrix}\Pi^{G}_{nm}(t,t') & \Pi^{F}_{nm}(t,t')\\ \Pi^{\overline{F}}_{nm}(t,t') & \Pi^{\overline{G}}_{nm}(t,t')\end{pmatrix}. \quad \text{(S12)}$$

Here again we separate out the Bogoliubov-Hartree-Fock contributions $\mathbf{\Sigma}^{HF}_{nm}$ from the second-order contributions describing collisions $\mathbf{\Sigma}_{nm}$.

The bare one-particle propagator and the Hartree-Fock self-energy is given by

$$\mathbf{G}^{-1}_{0,nm}(t,t') = \left[i\tau_3\frac{\partial}{\partial t}-\varepsilon_n\mathbb{1}\right]\delta_{nm}\delta(t-t'), \quad \text{(S13)}$$

The Hartree-Fock self-energies are 2×2 matrices in Bogoliubov space and depend only on one time argument $\mathbf{S}^{HF}(t,t')=\mathbf{S}^{HF}(t)\delta(t-t')$, $\mathbf{\Sigma}^{HF}(t,t')=\mathbf{\Sigma}^{HF}(t)\delta(t-t')$

$$\mathbf{S}^{HF}_{\alpha\beta}(t) = \begin{pmatrix}S^{HF}_{\alpha\beta}(t) & W^{HF}_{\alpha\beta}(t)\\ W^{HF}_{\alpha\beta}(t)^* & S^{HF}_{\alpha\beta}(t)^*\end{pmatrix}, \quad \text{(S14)}$$

$$\mathbf{\Sigma}^{HF}_{nm}(t) = \begin{pmatrix}\Sigma^{HF}_{nm}(t) & \Omega^{HF}_{nm}(t)\\ \Omega^{HF}_{nm}(t)^* & \Sigma^{HF}_{nm}(t)^*\end{pmatrix}. \quad \text{(S15)}$$

$\mathbf{S}^{HF}$ and $\mathbf{\Sigma}^{HF}$ contain contributions proportional to U, J' and K and describe the dynamical shift of the condensate and the single-particle levels due to the dynamical change of their occupation numbers and their interactions:

$$\mathbf{S}^{HF}_{\alpha\alpha}(t) = \frac{i}{2}U\,\mathrm{Tr}\left[\mathbf{C}_{\alpha\alpha}(t,t)\right]\mathbb{1} + i\frac{K}{2}\sum_{n,m=1}^{M}\left\{\frac{1}{2}\mathrm{Tr}\left[\mathbf{F}^{<}_{nm}(t,t)\right]\mathbb{1}+\mathbf{F}^{<}_{nm}(t,t)\right\}$$

$$\mathbf{S}^{HF}_{12}(t,t') = \mathbf{S}^{HF}_{21}(t,t') = i\frac{J'}{2}\sum_{n,m=1}^{M}\left\{\frac{1}{2}\mathrm{Tr}\left[\mathbf{F}^{<}_{nm}(t,t)\right]\mathbb{1}+\mathbf{F}^{<}_{nm}(t,t)\right\}, \quad \text{(S16)}$$

and

$$\mathbf{\Sigma}^{HF}_{nm}(t,t') = i\frac{K}{2}\sum_{\alpha}\left\{\mathbf{C}_{\alpha\alpha}(t,t)+\frac{1}{2}\mathrm{Tr}\left[\mathbf{C}_{\alpha\alpha}(t,t)\right]\mathbb{1}\right\} + i\frac{J'}{2}\sum_{\alpha\neq\beta}\left\{\mathbf{C}_{\alpha\beta}(t,t)+\frac{1}{2}\mathrm{Tr}\left[\mathbf{C}_{\alpha\beta}(t,t)\right]\mathbb{1}\right\} + iU'\sum_{\ell,s=1}^{M}\left\{\mathbf{F}_{\ell s}(t,t)+\frac{1}{2}\mathrm{Tr}\left[\mathbf{F}_{\ell s}(t,t)\right]\mathbb{1}\right\} \quad \text{(S17)}$$

where ε_n are equidistant eigenenergies of the trap, $\varepsilon_2=\varepsilon_1+\Delta$ and so on.

Evaluation of the integrals on the left-hand side of the Dyson equations gives

$$\left[i\tau_3\delta_{\alpha\gamma}\frac{\partial}{\partial t}-\mathbb{1}E_{\alpha\gamma}-\mathbf{S}^{HF}_{\alpha\gamma}(t)\right]\mathbf{C}_{\gamma\beta}(t,t') = -i\int_{-\infty}^{t} d\bar{t}\boldsymbol{\gamma}_{\alpha\gamma}(t,\bar{t})\mathbf{C}_{\gamma\beta}(\bar{t},t'), \quad \text{(S18)}$$

$$\left[i\tau_3\delta_{n\ell}\frac{\partial}{\partial t}-\varepsilon_n\delta_{n\ell}\mathbb{1}-\mathbf{\Sigma}^{HF}_{n\ell}(t)\right]\mathbf{A}_{\ell m}(t,t') = -i\int_{t'}^{t} d\bar{t}\,\mathbf{\Gamma}_{n\ell}(t,\bar{t})\mathbf{A}_{\ell m}(\bar{t},t'), \quad \text{(S19)}$$

$$\left[i\tau_3\delta_{n\ell}\frac{\partial}{\partial t}-\varepsilon_n\delta_{n\ell}\mathbb{1}-\mathbf{\Sigma}^{HF}_{n\ell}(t)\right]\mathbf{F}_{\ell m}(t,t') = -i\Big[\int_{-\infty}^{t} d\bar{t}\,\mathbf{\Gamma}_{n\ell}(t,\bar{t})\mathbf{F}_{\ell m}(\bar{t},t') - \int_{-\infty}^{t'} d\bar{t}\,\mathbf{\Pi}_{n\ell}(t,\bar{t})\mathbf{A}_{\ell m}(\bar{t},t')\Big]. \quad \text{(S20)}$$

Eqs. (S18), (S19) and (S20) constitute the general equations of motion for the condensate and the non-condensate (spectral and statistical) propagators. They are coupled via the self-energies which are functions of these propagators and must be evaluated self-consistently in order to obtain a conserving approximation. The higher order interaction terms on the right-hand side of the equations of motion describe inelastic quasi-particle collisions. They are, in general, responsible for quasi-particle damping, damping of the condensate oscillations and for thermalization. We consider them in detail in the next section.

II. EVALUATION OF THE SECOND ORDER CONTRIBUTIONS TO SELF-ENERGIES DUE TO COLLISIONS

The collisional terms are taken into accont within the full second order approximation Due to the symmetry relation (S8) we can express the collisional self-energies in terms of only upper left and upper right components of the spectral function $\mathbf{A}_{nm}(t,t')$ and the statistical function $\mathbf{F}_{nm}(t,t')$. We obtain for γ in (S18)

$$\begin{aligned}\gamma^G_{\alpha\alpha'}(t,t') &= R^2\sum_{nls}\sum_{n'l's'}\Big(\mathrm{F}^G_{nn'}(t,t')\Big\{4\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\\ &\qquad +2\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\\ &+\mathrm{A}^G_{nn'}(t,t')\Big\{4\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')+2\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\Big)\\ \gamma^F_{\alpha\alpha'}(t,t') &= R^2\sum_{nls}\sum_{n'l's'}\Big(\mathrm{F}^F_{nn'}(t,t')\Big\{4\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\\ &\qquad +2\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\\ &+\mathrm{A}^F_{nn'}(t,t')\Big\{4\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')+2\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\Big).\end{aligned} \tag{S21}$$

Here we introduced for the sake of shorter notations

$$\begin{aligned}\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t') &= \mathrm{A}^G_{\ell\ell'}(t,t')\mathrm{F}^F_{ss'}(t,t')+\mathrm{F}^G_{\ell\ell'}(t,t')\mathrm{A}^F_{ss'}(t,t')\\ \Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t') &= \mathrm{A}^G_{\ell\ell'}(t,t')\mathrm{F}^G_{ss'}(t,t')^*+\mathrm{F}^G_{\ell\ell'}(t,t')\mathrm{A}^G_{ss'}(t,t')^*\\ \Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t') &= \mathrm{F}^G_{\ell\ell'}(t,t')\mathrm{F}^F_{ss'}(t,t')-\frac{1}{4}\mathrm{A}^G_{\ell\ell'}(t,t')\mathrm{A}^F_{ss'}(t,t')\\ \Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t') &= \mathrm{F}^G_{\ell\ell'}(t,t')\mathrm{F}^G_{ss'}(t,t')^*-\frac{1}{4}\mathrm{A}^G_{\ell\ell'}(t,t')\mathrm{A}^G_{ss'}(t,t')^*\end{aligned}$$

and so on. The remaining γ-s are related to γ^G and γ^F by symmetry relations

$$\begin{aligned}\gamma^G(t,t')^* &= -\gamma^{\overline{G}}(t,t')=\gamma^G(t',t),\\ \gamma^F(t,t')^* &= -\gamma^{\overline{F}}(t,t')=\gamma^{\overline{F}}(t',t).\end{aligned} \tag{S22}$$

For $\mathbf{\Gamma}$ we get

$$\begin{aligned}\Gamma^G_{nn'}(t,t') &= 2iR^2\sum_{\alpha\ell s}\sum_{\alpha'\ell's'}\Big(2a^*_\alpha(t)a^*_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t')\\ &+a^*_\alpha(t)a_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}](t,t')\\ &-2a_\alpha(t)a_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{F}^*,\mathrm{G}](t,t')\\ &-2a_\alpha(t)a^*_{\alpha'}(t')\Big\{\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')+\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\Big)\\ &+(U')^2\sum_{m\ell s}\sum_{m'\ell's'}\Big(\mathrm{F}^G_{mm'}(t,t')\Big\{4\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\\ &+2\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\\ &+\mathrm{A}^G_{mm'}(t,t')\Big\{4\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')+2\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\Big)\end{aligned} \tag{S23}$$

and

$$\begin{aligned}\Gamma^F_{nn'}(t,t') &= 2iR^2\sum_{\alpha\ell s}\sum_{\alpha'\ell's'}\Big(2a^*_\alpha(t)a_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t')\\ &+a^*_\alpha(t)a^*_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}](t,t')\\ &-2a_\alpha(t)a^*_{\alpha'}(t')\Lambda^{\ell\ell'}_{ss'}[\mathrm{G}^*,\mathrm{F}](t,t')\\ &-2a_\alpha(t)a_{\alpha'}(t')\Big\{\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')+\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\Big)\\ &+(U')^2\sum_{m\ell s}\sum_{m'\ell's'}\Big(\mathrm{F}^F_{mm'}(t,t')\Big\{2\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\\ &+4\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\\ &+\mathrm{A}^F_{mm'}(t,t')\Big\{2\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')+4\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\Big).\end{aligned} \tag{S24}$$

Other self-energy parts in the Dyson equation for the statistical function have the following contributions

$$\begin{aligned}\Pi^G_{nn'}(t,t') &= 2iR^2\sum_{\alpha\ell s}\sum_{\alpha'\ell's'}\Big(2a^*_\alpha(t)a^*_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t')\\ &+a^*_\alpha(t)a_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}](t,t')\\ &-2a_\alpha(t)a_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{F}^*,\mathrm{G}](t,t')\\ &-2a_\alpha(t)a^*_{\alpha'}(t')\Big\{\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')+\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\Big)\\ &+(U')^2\sum_{m\ell s}\sum_{m'\ell's'}\Big(\mathrm{F}^G_{mm'}(t,t')\Big\{4\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\\ &+2\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\\ &-\frac{1}{2}\mathrm{A}^G_{mm'}(t,t')\Big\{2\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')+\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\Big)\end{aligned} \tag{S25}$$

and

$$\begin{aligned}\Pi^F_{nn'}(t,t') &= 2iR^2\sum_{\alpha\ell s}\sum_{\alpha'\ell's'}\Big(2a^*_\alpha(t)a_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{F}](t,t')\\ &+a^*_\alpha(t)a^*_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}](t,t')\\ &-2a_\alpha(t)a^*_{\alpha'}(t')\Xi^{\ell\ell'}_{ss'}[\mathrm{G}^*,\mathrm{F}](t,t')\\ &-2a_\alpha(t)a_{\alpha'}(t')\Big\{\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')+\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\Big\}\Big)\\ &+(U')^2\sum_{m\ell s}\sum_{m'\ell's'}\Big(\mathrm{F}^F_{mm'}(t,t')\Big\{2\Xi^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')\\ &+4\Xi^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\\ &-\frac{1}{2}\mathrm{A}^F_{mm'}(t,t')\Big\{\Lambda^{\ell\ell'}_{ss'}[\mathrm{F},\mathrm{F}^*](t,t')+2\Lambda^{\ell\ell'}_{ss'}[\mathrm{G},\mathrm{G}^*](t,t')\Big\}\Big).\end{aligned} \tag{S26}$$

The following symmetry relations apply here

$$\begin{aligned}\Gamma^G(t,t')^* &= -\Gamma^{\overline{G}}(t,t')=\Gamma^G(t',t),\\ \Gamma^F(t,t')^* &= -\Gamma^{\overline{F}}(t,t')=\Gamma^{\overline{F}}(t',t),\\ \Pi^G(t,t')^* &= -\Pi^{\overline{G}}(t,t')=-\Pi^G(t',t),\\ \Pi^F(t,t')^* &= -\Pi^{\overline{F}}(t,t')=-\Pi^{\overline{F}}(t',t).\end{aligned} \tag{S27}$$

Because of the symmetry relations (S8) we can express all quantities appearing in the equations of motion with the later time argument on the left side. Final equations of motion for the spectral function are

$$
\begin{aligned}
i\frac{\partial}{\partial t}\mathrm{A}^G_{nm}(t,t') &= (\varepsilon_n\delta_{n\ell} + \Sigma^{HF}_{n\ell})\mathrm{A}^G_{\ell m}(t,t') \\
&\quad -\Omega^{HF}_{n\ell}(t)(\mathrm{A}^F_{\ell m}(t,t'))^* \\
-i\int_{t'}^{t} d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{A}^G_{\ell m}(\bar{t},t') &+ \Gamma^F_{n\ell}(t,\bar{t})\mathrm{A}^{\overline{F}}_{\ell m}(\bar{t},t')\Big] \\
i\frac{\partial}{\partial t}\mathrm{A}^F_{nm}(t,t') &= (\varepsilon_n\delta_{n\ell} + \Sigma^{HF}_{n\ell})\mathrm{A}^F_{\ell m}(t,t') \\
&\quad -\Omega^{HF}_{n\ell}(t)(\mathrm{A}^G_{\ell m}(t,t'))^* \\
-i\int_{t'}^{t} d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{A}^F_{\ell m}(\bar{t},t') &+ \Gamma^F_{n\ell}(t,\bar{t})\mathrm{A}^{\overline{G}}_{\ell m}(\bar{t},t')\Big]\,. \qquad \text{(S28)}
\end{aligned}
$$

For the statistical function we get

$$
\begin{aligned}
i\frac{\partial}{\partial t}\mathrm{F}^G_{nm}(t,t') &= (\varepsilon_n\delta_{n\ell} + \Sigma^{HF}_{n\ell})\mathrm{F}^G_{\ell m}(t,t') \\
&\quad -\Omega^{HF}_{n\ell}(t)(\mathrm{F}^F_{\ell m}(t,t'))^* \\
-i\int_{0}^{t} d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{F}^G_{\ell m}(\bar{t},t') &+ \Gamma^F_{n\ell}(t,\bar{t})\mathrm{F}^{\overline{F}}_{\ell m}(\bar{t},t')\Big] \\
+i\int_{0}^{t} d\bar{t}\,\Big[\Pi^G_{n\ell}(t,\bar{t})\mathrm{A}^G_{\ell m}(\bar{t},t') &+ \Pi^F_{n\ell}(t,\bar{t})\mathrm{A}^{\overline{F}}_{\ell m}(\bar{t},t')\Big] \\
i\frac{\partial}{\partial t}\mathrm{F}^F_{nm}(t,t') &= (\varepsilon_n\delta_{n\ell} + \Sigma^{HF}_{n\ell})\mathrm{F}^F_{\ell m}(t,t') \\
&\quad -\Omega^{HF}_{n\ell}(t)(\mathrm{F}^G_{\ell m}(t,t'))^* \\
-i\int_{0}^{t} d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{F}^F_{\ell m}(\bar{t},t') &+ \Gamma^F_{n\ell}(t,\bar{t})\mathrm{F}^{\overline{G}}_{\ell m}(\bar{t},t')\Big] \\
+i\int_{0}^{t} d\bar{t}\,\Big[\Pi^G_{n\ell}(t,\bar{t})\mathrm{A}^F_{\ell m}(\bar{t},t') &+ \Pi^F_{n\ell}(t,\bar{t})\mathrm{A}^{\overline{G}}_{\ell m}(\bar{t},t')\Big] \qquad \text{(S29)}
\end{aligned}
$$

The equations (S28) and (S29) are coupled to the equations of motion for the condensates amplitudes

$$
\begin{aligned}
i\frac{\partial}{\partial t}a_\alpha(t) &= -Ja_{\beta\neq\alpha}(t) + S^{HF}_{\alpha\beta}a_\beta(t) + W^{HF}_{\alpha\beta}a^*_\beta(t) \\
&\quad -i\int_0^t d\bar{t}\,\left[\gamma^G_{\alpha\beta}(t,\bar{t})a_\beta(\bar{t}) + \gamma^F_{\alpha\beta}(t,\bar{t})a^*_\beta(\bar{t})\right]. \qquad \text{(S30)}
\end{aligned}
$$

Equations (S28), (S29) and (S30) are then solved numerically for various values of U, U', J and R.

III. NOTES ON NUMERICAL CALCULATIONS

In order to solve numerically our system of integro-differential equations, we discretize the two time arguments, t and t' with a constant time-step Δt (see Fig. S1). As a result, our spectral and statistical functions become matrices in the two-dimensional time plane. For

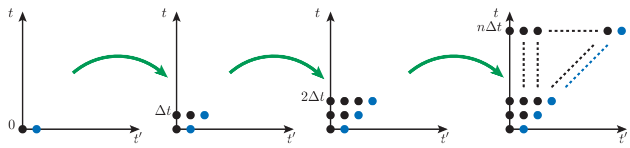

FIG. S1: Evolution of the time grid in the two-time plane.

instance,

$$
\mathrm{F}(t,t') = \begin{pmatrix}
\mathrm{F}(0,0) & \mathrm{F}(0,\Delta t) & \cdots & \mathrm{F}(0,n\Delta t) \\
\mathrm{F}(\Delta t,0) & \mathrm{F}(\Delta t,\Delta t) & \cdots & \mathrm{F}(\Delta t,n\Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
\mathrm{F}(n\Delta t,0) & \mathrm{F}(n\Delta t,\Delta t) & \cdots & \mathrm{F}(n\Delta t,n\Delta t)
\end{pmatrix} \qquad \text{(S31)}
$$

where both time arguments are counted from τ_c, which is a finite time-scale at which the nonequilibrium dynamics sets in [2], so that $\mathrm{F}^G(0,0) \equiv \mathrm{F}^G(\tau_c,\tau_c)$ etc. Both time scales go up to $t_{max} = n\Delta t$. In our case $t_{max} = 10$. Fortunately, due to the symmetry relations (S8) it is sufficient to calculate only half of the components of our propagators, i.e. the triangular matrix

$$
\mathrm{F}(t,t')_{tri} = \begin{pmatrix}
\mathrm{F}(0,0) & \mathrm{F}(0,\Delta t) & \cdots & \mathrm{F}(0,n\Delta t) \\
0 & \mathrm{F}(\Delta t,\Delta t) & \cdots & \mathrm{F}(\Delta t,n\Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathrm{F}(n\Delta t,n\Delta t),
\end{pmatrix} \qquad \text{(S32)}
$$

which is reflected in the time-plane grid in Fig.S1. The blue points on the grid constitute additional copies of the diagonal (propagators with equal time arguments) contributions, necessary to properly perform the fourth order Runge-Kutta method. Symmetry relations for the self-energies (S27) also contribute to simplifications, as we can rewrite all the integrands in Eqs. (S28), (S29) with time argument corresponding to later time on the left, e.g. time convolutions in the equation of motion for F^G can be rewritten as

$$
\begin{aligned}
&-i\int_0^t d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{F}^G_{\ell m}(\bar{t},t') + \Gamma^F_{n\ell}(t,\bar{t})\mathrm{F}^{\overline{F}}_{\ell m}(\bar{t},t')\Big] \\
&+i\int_0^t d\bar{t}\,\Big[\Pi^G_{n\ell}(t,\bar{t})\mathrm{A}^G_{\ell m}(\bar{t},t') + \Pi^F_{n\ell}(t,\bar{t})\mathrm{A}^{\overline{F}}_{\ell m}(\bar{t},t')\Big] \\
&= i\int_0^{t'} d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{F}^G_{\ell m}(t',\bar{t})^* + \Gamma^F_{n\ell}(t,\bar{t})\mathrm{F}^F_{\ell m}(t',\bar{t})^*\Big] \\
&-i\int_{t'}^t d\bar{t}\,\Big[\Gamma^G_{n\ell}(t,\bar{t})\mathrm{F}^G_{\ell m}(\bar{t},t') - \Gamma^F_{n\ell}(t,\bar{t})\mathrm{F}^F_{\ell m}(\bar{t},t')^*\Big] \\
&+i\int_0^{t'} d\bar{t}\,\Big[\Pi^G_{n\ell}(t,\bar{t})\mathrm{A}^G_{\ell m}(t',\bar{t})^* + \Pi^F_{n\ell}(t,\bar{t})\mathrm{A}^F_{\ell m}(t',\bar{t})^*\Big]
\end{aligned} \qquad \text{(S33)}
$$

In nonequilibrium it is conventional to introduce mixed or Wigner coordinates: $\tau = t - t'$ and $T = (t + t')/2$ and then Fourier transform spectral functions and statistical functions with respect to the relative coordinate. In this way one can extract information about spectrum and distribution function for different values of T and check if system approaches equilibrium with increasing T. In our case it is done by reading off the calculated spectral and statistical functions belonging to diagonals with the slope equal to -1 from the $t-t'$ plane in Fig. S1. Those will be data for fixed T-s. We then Fourier transform them with respect to τ. We checked the numerical accuracy by varying the time step dt used in the differential equation solver. All the results are reproducible and independent of dt.

IV. ENERGY OF THE BEC SUBSYSTEM

In the regimes where the BEC is effectively decoupled from the QP system, i.e., for $t < \tau_c$ as well as in the long-time limit, $t \gg \tau$, the total energy of the subsystem of BECs in the two wells can be calculated explicitly as the expectation value of the coherent parts of the Hamiltonian only, H_{coh} and H_J. Hence, the general expression for the BEC energy in these regimes is,

$$\begin{aligned} E_{BEC} &= \sum_{\alpha=1,2} \left[\varepsilon_0(N_\alpha) + \frac{U}{2}(N_\alpha - 1)N_\alpha \right] \\ &- 2J\sqrt{N_1 N_2} - 3J' N_{qp}\sqrt{N_1 N_2} \end{aligned} \tag{S34}$$

where N_1, N_2, N_{qp} are the particle numbers in the left and right potential wells and in the quasipartcle system, respectively. These numbers can be expressed in terms of the total particle number N, the total condensate number N_c, and the population imballance z as

$$N_1 + N_2 = N_c\,, \qquad N_1 - N_2 = zN_c\,, \qquad N_c + N_{qp} = N\ .$$

Inserting this in Eq. (S34), the BEC energy reads in terms of reduced interaction constants $u = NU/J$, $j' = NJ'/J$, and the condensate fraction $f = N_c/N$,

$$\begin{aligned} E_{BEC} &= \varepsilon_0 f N + \left[\frac{u}{4} f \left(f - \frac{2}{N} \right) + \frac{u}{4} z^2 f^2 \right. \\ &- \left. f\left(1 - \frac{3}{2} j'(1-f)\right)\sqrt{1-z^2} \right] NJ\ . \end{aligned} \tag{S35}$$

Hence, the initial-state energy at $t = 0$, where $\varepsilon_0 = 0$, $f = 1$, and $z = z(0) = z_0$, reads,

$$E_{BEC}(0) = \frac{u}{4}\left[(1 + z_0^2 - 2/N) - \sqrt{1 - z_0^2} \right] NJ\ . \tag{S36}$$

The final-state energy for $t \to \infty$, where $\varepsilon_0 = \varepsilon_0(\infty) \neq 0$ (renormalized by QP interactions), $f = f_\infty < 1$ (finite, but decoupled QP population), and $z = 0$ (BEC oscillations damped out), reads,

$$\begin{aligned} E_{BEC}(\infty) &= \left[f_\infty \frac{\varepsilon_0(\infty)}{J} + \frac{u}{4} f_\infty \left(f_\infty - \frac{2}{N} \right) \right. \\ &- \left. f_\infty \left(1 + \frac{3}{2} j'(1 - f_\infty) \right) \right] NJ\ . \end{aligned} \tag{S37}$$

The final-state parameters $\varepsilon_0(\infty)$ and f_∞ are obtained from the numerical solutions of the Keldysh-Bogoliubov equations of motion for the Green's functions, derived above.

[1] J. Rammer, *Quantum Field Theory of Non-equilibrium States*, Cambridge University Press (2007).

[2] M. Trujillo-Martinez, A. Posazhennikova, and J. Kroha, New. J. Phys. **17**, 013006 (2015).