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Physikalisches Institut
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http://www.kroha.uni-bonn.de/teaching

Special Topics in Condensed Matter Theory
Winter term 2016/17

Exercise 2

(Solutions due on 11 November, 2016)

2.1 Green's functions for noninteracting electrons (15 points)

In the lecture the position dependent Green's function was defined. In the same way one can define a momentum dependent retarded Green's function,

$$G^R_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\frac{1}{Z_G}\mathrm{tr}\left\{\mathrm{e}^{-\beta(H-\mu N)}\left[c_{\mathbf{k}\sigma}(t), c^\dagger_{\mathbf{k}\sigma}(t')\right]_+\right\} \tag{1}$$

The advanced and time-ordered momentum dependent Green's functions are defined in analogy to the lecture. Here we will consider a system of noninteracting electrons,

$$\mathcal{H}_0 = H_0 - \mu N = \sum_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}} - \mu)c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}$$

a) Determine the time-dependence of $c_{\mathbf{k}\sigma}(t)$ and $c^\dagger_{\mathbf{k}\sigma}(t')$ for the noninteracting system $\mathcal{H}_0$ by using the equation of motion for a Heisenberg operator.

b) Compute the retarded Green's function (1) for the noninteracting system using the results of b).

$$G^{(0)R}_{\mathbf{k}\sigma}(t,t') = -\mathrm{i}\Theta(t-t')\mathrm{e}^{-\mathrm{i}(\epsilon_{\mathbf{k}}-\mu)(t-t')} = G^{(0)R}_{\mathbf{k}\sigma}(t-t') \tag{2}$$

c) Derive the Fourier transform

$$G^{(0)R}_{\mathbf{k}\sigma}(\omega) = \int_{-\infty}^{\infty} d(t-t')\, G^{(0)R}_{\mathbf{k}\sigma}(t-t')\,\mathrm{e}^{\mathrm{i}\omega(t-t')} = \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + \mathrm{i}\eta}\ , \tag{3}$$

where $\eta > 0$ is an infinitesimal, positive number.

Hint: Observe and discuss that an infinitesimal imaginary part $i\eta$ must be introduced to the energy in order to make the integral convergent.

In an analogous manner, calculate the expression for the noninteracting, advanced Green's function $G^{(0)A}_{\mathbf{k}\sigma}(\omega)$.

d) The Green's function can also be derived from its equation of motion. Using the Heisenberg equation of motion for the operators $c_{\mathbf{k}\sigma}(t)$ or $c^\dagger_{\mathbf{k}\sigma}(t')$, show that the retarded as well as the advanced, noninteracting Green's function obeys the same equation of motion,

$$(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{H}_0)G^{(0)R/A}_{\mathbf{k}\sigma}(t-t') = \delta(t-t')\ ,$$

i.e. a "Schrödinger equation with $\delta-$Inhomogeneity".

e) Fourier transform the equation of motion of d) to energy space and solve it to obtain the retarded and advanced Green's functions in momentum and energy space $(\mathbf{k}, \omega)$. Discuss how the *boundary condition* of causality (retarded Green's function) and anticausality (advanced Green's function) is implemented in this energy-dependent Green's function [Hint: compare problem c)].

f) In the general, interacting case, the retarded Green's function contains a non-infinitesimal imaginary part in the denominator,

$$G^R_{\mathbf{k}\sigma}(\omega) = \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu) + \mathrm{i}/(2\tau)}$$

Use the residue theorem to calculate the time-dependent Green's function by Fourier transform. How does the Green's function behave for large $(t-t')$? Give a physical interpretation for τ and try to explain why a finite τ may occur.